Definition 1. Let T be a linear transformation from a finite dimensional vector space V to itself. Let $\mathfrak{B}$ be a basis of V. The $\mathfrak{B}$-*matrix* of the transformation T is the matrix B, which satisfies

$$B[f]_{\mathfrak{B}} = [T(f)]_{\mathfrak{B}}, \quad \text{for every } f \text{ in } V.$$

Theorem 2. *Let T be a linear transformation from a finite dimensional vector space V to itself, let $\mathfrak{B} = \{f_1, \ldots, f_n\}$ be a basis for V, and let B be the $\mathfrak{B}$-matrix for T. Then*

$$B = \begin{bmatrix} [T(f_1)]_{\mathfrak{B}} & \cdots & [T(f_n)]_{\mathfrak{B}} \end{bmatrix}.$$

Example 3. Let T be the linear transformation from P_2 (the set of polynomials of degree at most 2) to P_2 defined by: $T(f) = f' + f''$. Determine the $\mathfrak{B}$-matrix of T with respect to the basis $\mathfrak{B} = \{1, x, x^2\}$.

Example 4. Let $V = \text{span}\{\cos(x), \sin(x)\}$, and let $T(f) = 3f + 2f' - f''$ be a linear transformation from V to V. Find the $\mathfrak{B}$-matrix of T with respect to the basis $\mathfrak{B} = \{\cos(x), \sin(x)\}$.

Definition 5. Let V be an n-dimensional vector space and let $\mathfrak{A}$ and $\mathfrak{B}$ be two bases for V. The *change of basis matrix* from the basis $\mathfrak{B}$ to the basis $\mathfrak{A}$ is the matrix S (sometimes denoted $S_{\mathfrak{B}\to\mathfrak{A}}$), which satisfies:

$$S[f]_{\mathfrak{B}} = [f]_{\mathfrak{A}}, \quad \text{for every } f \text{ in } V.$$

Moreover, if $\mathfrak{B} = \{b_1, \ldots, b_n\}$, then

$$S = \begin{bmatrix} [b_1]_{\mathfrak{A}} & \cdots & [b_n]_{\mathfrak{A}} \end{bmatrix}.$$

Example 6. Let $V = \text{span}\{e^x, e^{-x}\}$, $\mathfrak{A} = \{e^x, e^{-x}\}$, and $\mathfrak{B} = \{e^x + e^{-x}, e^x - e^{-x}\}$. Compute the change of basis matrix $S_{\mathfrak{B}\to\mathfrak{A}}$.

Theorem 7. *Let V be a finite dimensional vector space with bases $\mathfrak{A}$ and $\mathfrak{B}$. Let S be the change of basis matrix from $\mathfrak{B}$ to $\mathfrak{A}$. Let T be a linear transformation from V to V, and let A and B be the $\mathfrak{A}$- and $\mathfrak{B}$-matrix of T, respectively. Then A is similar to B, and $A = SBS^{-1}$.*