Gram-Schmidt
University of Massachusetts Amherst
Math 235 — Spring 2014

Definition 1. The *length* of a vector v is equal to $\sqrt{v \cdot v}$. The length of v is denoted by $|v|$.

Definition 2. A vector u is a *unit vector* if the length of the vector is 1. Any vector v may be scaled to length 1 by dividing by the length of the vector:

$$\frac{v}{|v|} \quad \text{is a unit vector in the direction of } v.$$

Definition 3. Two vectors v and u are orthogonal if and only if $v \cdot u = 0$.

Definition 4. A basis $\mathfrak{B} = \{e_1, \ldots, e_n\}$ is *orthonormal* if every vector in the basis is a unit vector and $e_i \cdot e_j = 0$ whenever $i \neq j$. That is, each vector in the basis is orthogonal to every other vector in the basis.

Theorem 5. *Given a basis $\mathfrak{B} = \{v_1, \ldots, v_n\}$, one can obtain an orthonormal basis $\mathfrak{O} = \{e_1, \ldots, e_n\}$ using* Gram-Schmidt orthogonalization*.*

The Gram-Schmidt process is as follows:

$$\begin{aligned}
u_1 &= v_1 \\
u_2 &= v_2 - \text{proj}_{u_1}(v_2) \\
u_3 &= v_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3) \\
u_4 &= v_4 - \text{proj}_{u_1}(v_4) - \text{proj}_{u_2}(v_3) - \text{proj}_{u_3}(v_4) \\
&\vdots \\
u_n &= v_n - \text{proj}_{u_1}(v_n) - \text{proj}_{u_2}(v_n) - \cdots - \text{proj}_{u_{n-1}}(v_n),
\end{aligned}$$

where

$$\text{proj}_u(v) = \frac{u \cdot v}{u \cdot u} u.$$

The vectors $u_1, \ldots, u_n$ form an *orthogonal basis*, meaning that these vectors are a basis for $\mathbb{R}^n$, and each vector is orthogonal to every other vector in the basis. To obtain an orthonormal basis, we need to rescale:

$$e_1 = \frac{u_1}{|u_1|}, \qquad e_2 = \frac{u_2}{|u_2|}, \qquad \cdots \qquad e_n = \frac{u_n}{|u_n|}.$$

Example 6. Find an orthonormal basis for the solution space of $x_1 + x_2 + x_3 + x_4 = 0$.

ANSWER: The vectors that satisfy this equation are the vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so a basis for the solution space is $\mathfrak{B} = \{v_1, v_2, v_3\}$, where

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Applying Gram-Schmidt, we get

$$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1 \end{bmatrix}$$

You may verify that $u_1 \cdot u_2 = u_1 \cdot u_3 = u_2 \cdot u_3 = 0$. Normalizing these vectors, we obtain an orthonormal basis:

$$e_1 = \frac{1}{\sqrt{2}} u_1 \qquad e_2 = \frac{1}{\sqrt{3/2}} u_2 \qquad e_3 = \frac{1}{\sqrt{4/3}} u_3.$$