7.3 — Eigenvectors
University of Massachusetts Amherst
Math 235 — Spring 2014

Recall that $\vec{v}$ is an *eigenvector* of an $n \times n$ matrix A if $A\vec{v} = \lambda\vec{v}$, where λ is an eigenvalue of A. The eigenvectors associated with the eigenvalue λ are the solutions to

$$(A - \lambda I)\vec{v} = \vec{0}.$$

In other words, the set of eigenvectors associated with λ is the kernel of the matrix $(A - \lambda I)$.

Definition 1. The *eigenspace* associated with λ is the kernel of the matrix $(A - \lambda I)$ and is denoted by E_λ:

$$E_\lambda = \ker(A - \lambda I).$$

Example 2. Determine the eigenspaces of the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

ANSWER: We computed the eigenvalues of this matrix in the previous set of notes by finding the roots of $\det(A - \lambda I)$. Here's another method to obtain the eigenvalues: in Example 15 in the 7.2 notes, we found that the characteristic polynomial of a 2×2 matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\lambda^2 - \operatorname{tr}(B)\lambda + \det(B) = \lambda^2 - (a + d)\lambda + (ad - bc).$$

Thus the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, is

$$\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1),$$

and the eigenvalues of A are -1 and 5. Now that we know the eigenvalues, we can compute the eigenspace.

For $\lambda = -1$, we have

$$E_{-1} = \ker(A + I) = \ker \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The rows of this last matrix correspond to the equation $x_1 + x_2 = 0$, hence a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in the kernel of the matrix if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. That is, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in the kernel if and only if $x_1 = -x_2$. Thus

$$E_{-1} = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

For $\lambda = 5$, we have

$$E_5 = \ker(A - 5I) = \ker \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}.$$

Here, a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in the kernel if and only if $x_1 = -x_2/2$. That is, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$, so

$$E_5 = \operatorname{span}\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}.$$

Example 3. Determine the eigenspaces of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

ANSWER: A is an upper triangular matrix, so its eigenvalues are on its diagonal: the eigenvalues are 0 (with algebraic multiplicity 1), and 1 (with algebraic multiplicity 2).

For $\lambda = 0$,

$$E_0 = \ker(A) = \ker \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in the if and only if $x_1 = -x_2$, and $x_3 = 0$. Hence

$$E_0 = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

For $\lambda = 1$,

$$E_1 = \ker(A - I) = \ker \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in the if and only if $x_2 = x_3 = 0$ (x_1 may be any value). Hence

$$E_0 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Definition 4. The *geometric multiplicity* of an eigenvalue λ is the dimension of its eigenspace E_λ.

Theorem 5. *If λ is an eigenvalue of A, then the algebraic multiplicity of λ is greater than or equal to the geometric multiplicity of λ.*

$$1 \leq (\textit{geom. multi. of } \lambda) \leq (\textit{alg. multi. of } \lambda)$$

Definition 6. Let A be an $n \times n$ matrix. A basis of $\mathbb{R}^n$ consisting of eigenvectors of A is called an eigenbasis of A.

Theorem 7. *If $E_1, \ldots, E_m$ are distinct eigenspaces of A, and we select one nonzero vector from each space: $\vec{v}_1 \in E_1, \vec{v}_2 \in E_2, \ldots, \vec{v}_m \in E_m$, then the set of vectors $\{v_1, \ldots, v_m\}$ is linearly independent.*

Theorem 8. *If A is an $n \times n$ matrix and A has n distinct eigenvalues, then there is an eigenbasis for A. (Just take one nonzero vector from each eigenspace.)*

Example 9. For each matrix, (i) find all its eigenvalues, (ii) find a basis for each eigenspace, and (iii) find an eigenbasis, if you can.

$$a) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \qquad b) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

ANSWER: a) Since this matrix is triangular, the eigenvalues are on the diagonal: the eigenvalues are 1, 2, and 3.

$$E_1 = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$E_2 = \ker \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$E_3 = \ker \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

We can take a vector from each eigenspace to obtain an eigenbasis for $\mathbb{R}^3$: $\mathfrak{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

b) There is only one eigenvalue: 1, and

$$E_1 = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

There is only one vector, and we need three linearly independent vectors to obtain a basis for $\mathbb{R}^3$. There is no eigenbasis in this case.

Theorem 10. *Suppose that A and B are similar.*

(a) A and B have the same characteristic polynomial.
(b) $\operatorname{rank}(A) = \operatorname{rank}(B)$, and $\operatorname{nullity}(A) = \operatorname{nullity}(B)$.
(c) A and B have the same eigenvalues with the same algebraic and geometric multiplicities. Their eigenvectors may be different.
(d) $\det(A) = \det(B)$, and $\operatorname{tr}(A) = \operatorname{tr}(B)$.