Day 36

#1 a) Wrong. When showing divergence w/ Direct comparison test you need the unknown series to be bigger than the diverging known series $\sum 1/n$

b) Wrong. $\lim_{n\to\infty} \frac{1}{\arctan n} = \frac{1}{\pi/2} = \frac{2}{\pi} \neq 0.$

#2 $\sum \frac{k^4}{2^k}$ ... terms are positive - Ratio Test

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^4}{2^{k+1}} \cdot \frac{2^k}{k} = \lim_{k\to\infty} \frac{(k+1)^4}{k^4} \cdot \frac{1}{2} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^4 \cdot \frac{1}{2} = 1 \cdot \frac{1}{2}$$

$r = 1/2 < 1$, so the series converges by the ratio test

#3a) $\sum_{k=1}^{\infty} \left(\frac{k+1}{2k}\right)^k$ ... terms are positive ... Root Test

$$\lim_{k\to\infty} \sqrt[k]{\left(\frac{k+1}{2k}\right)^k} = \lim_{k\to\infty} \frac{k+1}{2k} \overset{HP}{=} \frac{1}{2} < 1$$ The series converges by the Root Test $(r<1)$

b) $\sum_{k=1}^{\infty} \left(1 + 3/k\right)^{k^2}$ ... terms are positive ... powers ... Root Test

$$\lim_{k\to\infty} \sqrt[k]{(1+3/k)^{k^2}} = \lim_{k\to\infty} \left(1 + 3/k\right)^{k^2/k} = \lim_{k\to\infty} \left(1+3/k\right)^k = e^3 > 1$$

The series diverges by the Root test $(r > 1)$

c) $\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)}\right)^k$ ... positive terms ... powers ... Root Test

$$\lim_{k\to\infty} \sqrt[k]{\left(\frac{1}{\ln(k+1)}\right)^k} = \lim_{k\to\infty} \frac{1}{\ln(k+1)} = 0 < 1$$. The series converges by the Root Test

#4a) $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ --- Factorials ... Use Ratio Test ... positive terms

$$\lim_{k\to\infty} \frac{((k+1)!)^2}{(2k+2)!} \cdot \frac{(2k)!}{(k!)^2} = \lim_{k\to\infty} \frac{(k+1)^2}{(2k+2)(2k+1)} = \lim_{k\to\infty} \frac{k^2+2k+1}{4k^2+6k+2} \overset{HP}{=} \frac{1}{4} < 1$$

since $r<1$, the series converges

4(b) $\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$ .. limit comparison w/ $\sum 1/k^2$ (or direct)

The terms are positive

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^2 \ln n} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$$

Since $\sum \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$), so does $\sum \frac{1}{k^2 \ln k}$

by limit comparison test

4c) $\sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$ ... positive terms ... factorial ... Ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = \lim_{n\to\infty} \frac{2\cancel{(n+1)}\, n^n}{(n+1)^{\cancel{n+1}}} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} 2\left(\frac{1}{1+1/n}\right)^n = \frac{2}{e} < 1.$$ By the ratio test, the series converges

#5 a) $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2+10}$. Alternating $a_n = \frac{1}{n^2+10} > 0$ (positive)

① $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2+10} = 0$ check ✓

② $\{a_n\}$ decreasing. Yes ... $f(x) = \frac{1}{x^2+10}$, $f'(x) = -1(x^2+10)^{-2}(2x) < 0$ for $x \geq 0$

$\therefore$ The series converges by the alternating series test.

b) $\sum_{k=2}^{\infty} (-1)^k \frac{\ln k}{k^2}$ ... Alternates $a_n = \frac{\ln n}{n^2} > 0$ -- positive terms

① $\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{\ln x}{x^2} \overset{L'H}{=} \lim_{x\to\infty} \frac{\overset{\to 0}{\frac{1}{x}}}{\underset{\to\infty}{2x}} = 0$ ✓

② Decreasing? $f(x) = \frac{\ln x}{x^2}$; $f'(x) = \frac{1/x\,(x^2) - 2x \ln x}{x^4} = \frac{1 - 2\ln x}{x^3} < 0$ ✓ checks for $x \geq 2$

$\therefore$ the series converges by the Alternating Series Test

c) $\sum_{k=1}^{\infty} \cos(k\pi) \frac{k^2}{2k^2+3} = \sum_{k=1}^{\infty} (-1)^k \frac{k^2}{2k^2+3}$ ... Alternating $a_n = \frac{n^2}{2n^2+3}$ ... positive terms

① $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2n^2+3} \overset{HP}{=} \frac{1}{2} \neq 0$

By the n^{th} term test for divergence
since $\lim_{n\to\infty} a_n \neq 0$, the series diverges