Math 131 Day 35: Practest 3

Part 1

For basic practice problems review Labs 9–11. Problems 1–4 apply our new techniques of integration to earlier applications.

1. Find the volume of the infinitely long solid region generated when the area in the first quadrant enclosed by $y = \sqrt{\dfrac{8}{x^2+6x+5}}$, from $x = 0$ to $x = \infty$ is revolved around the x-axis. Use disks.

2. Find the average value of $f(x) = \frac{1}{x^2-9x+20}$ on the interval $[1, 3]$.

3. Find the volume of the solid region generated when the area enclosed by $y = \sqrt{\dfrac{8}{x^2+6x+5}}$, from $x = -1$ to $x = 0$ is revolved around the x-axis. Use disks.

4. a) Find the area in the first quadrant enclosed by $y = \frac{1}{\sqrt{x^2-9}}$ $y = 0$, and $x = 3$ and $x = 5$.

b) Find the area in the first quadrant enclosed by $y = \frac{1}{\sqrt{x^2-9}}$ $y = 0$, and $x = 5$ and $x = \infty$.

Part 2: Techniques

1. a) Someone takes a maintenance medication: 48 mg once every 24 hr. Every 24 hr one-half of the drug is **eliminated** from the blood stream. Find the recurrence relation for the sequence $\{d_n\}$ where d_n is the amount of the drug in the bloodstream immediately after dose n.

b) Write out the first four terms of the sequence. Does the sequence appear to be monotonic?

c) Find the limit L of the sequence.

2. Try these; many are similar looking integrals. The first few are improper.

a) $\displaystyle\int_0^\infty \frac{4}{4+x^2}\,dx$ **b)** $\displaystyle\int_3^\infty \frac{4}{4-x^2}\,dx$ **c)** $\displaystyle\int_1^2 \frac{4}{4-x^2}\,dx$ **d)** $\displaystyle\int_3^\infty \frac{4x}{4-x^2}\,dx$

e) $\displaystyle\int \frac{4x+1}{x^2-5x+4}\,dx$ **f)** $\displaystyle\int \frac{4x+8}{x^2+4x+5}\,dx$

g) $\displaystyle\int \frac{-4x+4}{(x-2)^2x}\,dx$ **h)** $\displaystyle\int \frac{4x+1}{x^2-4}\,dx$ **i)** $\displaystyle\int \frac{4}{(4-x)^{2/3}}\,dx$

j) $\displaystyle\int \frac{8x+4}{x^3+x^2-2x}\,dx$ **k)** $\displaystyle\int_0^2 \frac{2x+6}{x^2+2x-8}\,dx$

l) $\displaystyle\int_4^\infty \frac{-3}{x^2-3x}$ **m)** $\displaystyle\int \frac{4x^2+8x+2}{x(x+1)^2}\,dx$ **n)** $\displaystyle\int \frac{4x}{(x+1)^3}\,dx$

3. Use L'Hopital's Rule if appropriate. (Answers **not** in order: 0, 0, 0, $\frac{1}{2}$, 1, 1 ln 2, 2, e, 3, 4, 5, -6, e^7.)

a) $\displaystyle\lim_{x\to1} \frac{x-1}{\ln x}$ **b)** $\displaystyle\lim_{x\to0} \frac{x^2+4x}{\sin 2x}$ **c)** $\displaystyle\lim_{x\to\infty} \frac{6x^2+3x-1}{2x^2+x}$

d) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{e^x}$ **e)** $\displaystyle\lim_{x\to1} \frac{6x+10}{3x+1}$ **f)** $\displaystyle\lim_{x\to0} \frac{\tan 5x}{\arcsin x}$ **g)** $\displaystyle\lim_{x\to\infty} \left(1+\tfrac{7}{x}\right)^x$

h) $\displaystyle\lim_{x\to0} \frac{\cos 4x - \cos 2x}{x^2}$ **i)** $\displaystyle\lim_{x\to0} \frac{e^x-1-x}{x^2}$ **j)** $\displaystyle\lim_{x\to0^+} 2x\ln x$ **k)** $\displaystyle\lim_{x\to\infty} x^2e^{-x}$

l) $\displaystyle\lim_{x\to\infty} \ln(2x+9)-\ln(x+7)$ **m)** $\displaystyle\lim_{x\to0^+} (2x)^x$ **n)** $\displaystyle\lim_{x\to0^+} (1+x)^{1/x}$

4. Find the limits of these sequences. Use the key limits when possible (indicate when you do so). For the last part, use the derivative formula: $\frac{d}{dx}(a^x) = a^x \ln a$, when $a > 0$.

a) $\displaystyle\left\{\left(1+\frac{3}{n}\right)^n\right\}_{n=1}^\infty$ **b)** $\displaystyle\{\ln(2n^2+7)-\ln(5n^2+n)\}_{n=1}^\infty$ **c)** $\displaystyle\left\{\frac{2\ln(n+1)}{n^2}\right\}_{n=1}^\infty$

d) $\displaystyle\left\{\left(\frac{2}{3}\right)^n\right\}_{n=1}^\infty$ **e)** $\displaystyle\left\{\left(\frac{-3}{2}\right)^n\right\}_{n=1}^\infty$ **f)** $\displaystyle\left\{\frac{4n^2-3n+1}{5n^2+7}\right\}_{n=1}^\infty$

g) $\displaystyle\left\{\left(\frac{n^2}{3^n}\right)\right\}_{n=1}^\infty$

5. Find the sums of these series, if they exist. Note the starting indices!

a) $\displaystyle\sum_{n=0}^{\infty} 4\left(\frac{-2}{5}\right)^n$ **b)** $\displaystyle\sum_{n=0}^{\infty} 4\left(\frac{10}{9}\right)^n$ **c)** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+n}$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{6}{n^2+7n+12}$ **e)** $\displaystyle\sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^n$ **f)** $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{-2}{3}\right)^n$

6. Determine whether the following series converge. First determine which test to use: nth term test, p-series test, integral test, geometric series test, or the comparison tests. Your final answer should consist of a little 'argument' (a sentence or two) and any necessary calculations. **Use appropriate mathematical language**.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^7}}$ **b)** $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{1+e^n}$ **c)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{16+9n^2}$ **d)** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{n^2}$

e) $\displaystyle\sum_{n=1}^{\infty} \ln(3n+3)-\ln(6n+2)$ **f)** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$ **g)** $\displaystyle\sum_{n=1}^{\infty} 2\left(\frac{-3}{11}\right)^n$ **h)** $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$

i) $\displaystyle\sum_{n=1}^{\infty} 6\left(\frac{5}{4}\right)^n$ **j)** $\displaystyle\sum_{n=0}^{\infty} \frac{3n^2}{n^3+1}$ **k)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)!}{(2n-1)!}$ **l)** $\displaystyle\sum_{n=2}^{\infty} \frac{3}{n^2+5n+4}$

m) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ **n)** $\displaystyle\sum_{k=1}^{\infty} \frac{2k}{e^{k^2}}$ **o)** $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^2+4k+3}$ **p)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

Brief Answers to Part 1. More complete answers are available on line.

1. $\frac{1}{2}(\ln 3+\ln 2-\ln 4)$.

2. Ans: $2\pi\ln 5$.

3. Improper (at -1). Ans: Diverges.

4. a) $\ln 3$
b) Ans: Diverges

Some Brief Answers to Part 2. More complete answers are available on line.

1. a) $d_1=48$ and $d_{n+1}=48+\frac{1}{2}d_n$. (b) $\{d_n\}=\{72,84,88,90,\ldots\}$. (c) $L=96$ mg.

2. Watch for typos. If you find any, let me know. The indefinite integrals are all $+c$.

a) π **b)** $-\ln 5$ **c)** diverges **d)** diverges

e) $\frac{17}{3}\ln|x-4|-\frac{5}{3}\ln|x-1|$ **f)** $2\ln|x^2+4x+5|$

g) $\dfrac{2}{x-2}+\ln\left|\dfrac{x}{x-2}\right|$ **h)** $2\sqrt{3}-2\pi/3$

i) $\frac{9}{4}\ln|x-2|+\frac{7}{4}\ln|x+2|$ **j)** $4\ln|x-1|-2\ln|x|-2\ln|x+2|$ **k)** $-\infty$ Diverges

l) $\lim_{b\to\infty}\ln\left|\frac{x}{x-3}\right|\Big|_4^b=-\ln|4|$ **m)** $2\ln|x|-\dfrac{2}{x+1}+2\ln|x+1|$ **n)** $-4(x+1)^{-1}+2(x+1)^{-2}$

4. **a)** e^3 **b)** $\ln\dfrac{4}{10}$ **c)** 0 **d)** 0 **e)** diverges **f)** $\dfrac{4}{5}$ **g)** 0

5. **a)** $\dfrac{20}{7}$ **b)** diverges **c)** 2 **d)** 2 **e)** $\dfrac{1}{4}$ **f)** $-\dfrac{6}{5}$

6. Those that converge: a, c, g, l, n, o, p. You should give an argument to justify your answers for these. The correct answer without justification will lose most of the points. (See the answers on line.)

Math 131 Practest 3: Answer Sketches

Part 1

1. $AV = \frac{1}{3-1}\int_1^3 \frac{1}{x^2-9x+20}\,dx = \int_1^3 \frac{1/2}{x-5} - \frac{1/2}{x-4}\,dx = \frac{1}{2}(\ln|x-5| - \ln|x-4|)\Big|_1^3 = \frac{1}{2}(\ln 2 - \ln 4 + \ln 3)$.

2. $\frac{8}{x^2+6x+5} = \frac{2}{x+1} - \frac{2}{x+5}$. So $\int_0^\infty \frac{8}{x^2+6x+5}\,dx = \int \frac{2}{x+1} - \frac{2}{x+5}\,dx = 2\pi\ln|\frac{x+1}{x+5}|$.

$$V = \pi\int_0^\infty \frac{8}{x^2+6x+5}\,dx = \lim_{b\to\infty} 2\pi\ln|\tfrac{x+1}{x+5}|\Big|_0^b = \lim_{b\to\infty} 2\pi[\ln|\tfrac{b+1}{b+5}| - \ln|\tfrac{1}{5}|] = 2\pi[\ln|1| - \ln|\tfrac{1}{5}|] = 2\pi\ln 5.$$

3. As above:

$$V = \pi\int_{-1}^0 \frac{8}{x^2+6x+5}\,dx = \lim_{a\to -1^+} 2\pi\ln\left|\frac{x+1}{x+5}\right|\Bigg|_a^0 = \lim_{a\to -1^+} 2\pi\left[\ln\left|\frac{1}{5}\right| - \ln\left|\frac{a+1}{a+5}\right|\right] = 2\pi\left[\ln\left|\frac{1}{5}\right| - \infty\right].\ \text{Diverges.}$$

4. a) Triangles. $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|$$

$$\int_3^5 \frac{1}{\sqrt{x^2-9}}\,dx = \lim_{a\to 3^+} \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\Bigg|_a^5 = \lim_{a\to 3^+} \ln\left|\frac{5}{3} + \frac{4}{3}\right| - \ln\left|\frac{a}{3} + \frac{\sqrt{a^2-9}}{3}\right| = \ln|3| - \ln|1| = \ln 3.$$

b) Use the work above.

$$\int_5^\infty \frac{1}{\sqrt{x^2-9}}\,dx = \lim_{b\to\infty} \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\Bigg|_5^b = \lim_{b\to\infty} \ln\left|\frac{b}{3} + \frac{\sqrt{b^2-9}}{3}\right| - \ln\left|\frac{5}{3} + \frac{4}{3}\right| = \infty - \ln|3|.\ \text{Diverges}$$

Part 2

1. a) $d_1 = 48$ and $d_{n+1} = 48 + \frac{1}{2}d_n$.

b) $\{d_n\} = \{72, 84, 88, 90, \ldots\}$.

c) Let $\lim\limits_{n\to\infty} d_n = L$. Now $d_{n+1} = 48 + \frac{1}{2}d_n$. So $\lim\limits_{n\to\infty} d_{n+1} = \lim\limits_{n\to\infty} 48 + \frac{1}{2}d_n$ or $L = 48 + \frac{1}{2}L$. So $\frac{1}{2}L = 48$, so $L = 96$ mg.

2. a) $\lim_{b\to\infty} 4\int_0^b \frac{1}{2^2+x^2}\,dx = \lim_{b\to\infty} 4\cdot\frac{1}{2}\arctan\frac{x}{2}\Big|_0^b = \lim_{b\to\infty} 2\arctan\frac{b}{2} - 0 = 2(\frac{\pi}{2}) - 0 = \pi$.

b) $\lim_{b\to\infty}\int_3^\infty \frac{4}{4-x^2}\,dx = \lim_{b\to\infty}\int_3^b \frac{1}{2-x} + \frac{1}{2+x}\,dx = \lim_{b\to\infty}(-\ln|2-x| + \ln|2+x|)\Big|_3^b = \lim_{b\to\infty}\ln|\frac{2+x}{2-x}|\Big|_3^b = \lim_{b\to\infty}\ln|\frac{2+b}{2-b}| - \ln 5 = \ln 1 - \ln 5 = -\ln 5$.

c) $\lim_{b\to 2^-}\int_1^b \frac{4}{4-x^2}\,dx = \lim_{b\to 2^-}\ln|\frac{2+x}{2-x}|\Big|_1^b = \lim_{b\to 2^-}\ln|\frac{2+b}{2-b}| - \ln 3 = +\infty$. Diverges.

d) u-sub: $\lim_{b\to\infty}\int_3^b \frac{4x}{4-x^2}\,dx = \lim_{b\to\infty}(-2\ln|4-x^2|)\Big|_3^b = \lim_{b\to\infty}\ln|4-b^2| - \ln 5 = +\infty$. Diverges.

e) $\int \frac{4x+1}{x^2-5x+4}\,dx = \int \frac{17/3}{x-4} - \frac{5/3}{x-1}\,dx = \frac{17}{3}\ln|x-4| - \frac{5}{3}\ln|x-1| + c$.

f) Not partial fractions. Can't factor. u-substitution: $u = x^2+4x+5$, $du = 2x+4\,dx$. $\int \frac{4x+8}{x^2+4x+5}\,dx = \int \frac{2}{u}\,du = 2\ln|u| + c = 2\ln|x^2+4x+5| + c$.

g) $\int -\frac{1}{x-2} - \frac{2}{(x-2)^2} + \frac{1}{x}\,dx = -\ln|x-2| + 2(x-2)^{-1} + \ln|x| + c$.

h) $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-4} = 2\tan\theta$. $x = 2 = 2\sec\theta \Rightarrow \theta = 0$. $x = 4 = 2\sec\theta \Rightarrow \theta = \pi/3$. $\int_0^{\pi/3}\frac{2\tan\theta}{2\sec\theta}2\sec\theta\tan\theta\,d\theta = 2\int_0^{\pi/3}\tan^2\theta\,d\theta = 2\int_0^{\pi/3}\sec^2\theta - 1\,d\theta = 2\tan\theta - 2\theta\Big|_0^{\pi/3} = 2\sqrt{3} - \frac{2}{3}\pi$.

i) $\int \frac{4x+1}{x^2-4}\,dx = \int \frac{9/4}{x-2} + \frac{7/4}{x+2}\,dx = \frac{9}{4}\ln|x-2| + \frac{7}{4}\ln|x+2| + c$.

j) $\int \frac{4}{x-1} - \frac{2}{x} - \frac{2}{x+2}\,dx = 4\ln|x-1| - 2\ln|x| - 2\ln|x+2| + c$.

k) $\int \frac{2x+6}{x^2+2x-8}\,dx = \int \frac{5/3}{x-2} + \frac{1/3}{x+4}\,dx = \frac{5}{3}\ln|x-2| + \frac{1}{3}\ln|x+4|$. So

$$\int_0^2 \frac{2x+6}{x^2+2x-8}\,dx = \lim_{b\to 2^-}\frac{5}{3}\ln|x-2| + \frac{1}{3}\ln|x+4|\Bigg|_0^b = \lim_{b\to 2^-}\frac{5}{3}\ln|b-2| + \frac{1}{3}\ln|b+4| - \left[\frac{5}{3}\ln|-2| + \frac{1}{3}\ln|4|\right]$$

$$= -\infty + \frac{1}{3}\ln|6| - \left[\frac{5}{3}\ln|-2| + \frac{1}{3}\ln|4|\right]. \text{ Diverges}$$

l) $\int \frac{-3}{x^2-3x}\,dx = \int \frac{1}{x} - \frac{1}{x-3}\,dx = \ln|x| - \ln|x-3| = \ln|\frac{x}{x-3}|$. So

$$\int_4^\infty \frac{3}{x^2-3x}\,dx = \lim_{b\to\infty} \ln\left|\frac{x}{x-3}\right|\Bigg|_4^b = \lim_{b\to\infty} \ln\left|\frac{b}{b-3}\right| - \ln|4| = \ln|1| - \ln|4| = -\ln|4|.$$

m) $\int \frac{4x^2+8x+2}{x(x+1)^2}\,dx = \int \frac{2}{x} + \frac{2}{x+1} + \frac{2}{(x+1)^2}\,dx = 2\ln|x| + 2\ln|x+1| - 2(x+1)^{-1} + c.$

n) $\dfrac{4x}{(x+1)^3} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} = \dfrac{Ax^2+2Ax+A+Bx-B+C}{(x+1)^3}.$

$$\begin{array}{llll} x^2: & A & & = 0 \\ x: & 2A+ & B & = 4 \\ \text{const:} & & B+ \quad C & = 0 \end{array} \Rightarrow A=0,\ B=4,\ C=-4. \quad \int \frac{4}{(x+1)^2} - \frac{4}{(x+1)^3}\,dx = -4(x+1)^{-1} + 2(x+1)^{-2} + c.$$

3. a) $\lim_{x\to1} \frac{x-1}{\ln x} \overset{\text{l'Ho}}{=} \lim_{x\to1} \frac{1}{1/x} = 1$

b) $\lim_{x\to0} \frac{x^2+4x}{\sin 2x} \overset{\text{l'Ho}}{=} \lim_{x\to0} \frac{2x+4}{2\cos 2x} = \frac{4}{2} = 2$

c) $\lim_{x\to\infty} \frac{6x^2+3x-1}{2x^2+x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{12x+3}{4x+1} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{12}{4} = 3$

d) $\lim_{x\to\infty} \frac{\ln x}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{1/x}{e^x} = 0$

e) $\lim_{x\to1} \frac{6x+10}{3x+1} = \frac{16}{4} = 4$

f) $\lim_{x\to0} \frac{\tan 5x}{\arcsin x} \overset{\text{l'Ho}}{=} \lim_{x\to0} \frac{5\sec^2 5x}{\frac{1}{\sqrt{1-x^2}}} = 5$

g) Recognize as a key limit: e^7. OR: assume $y = \lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x$.

$$\ln y = \ln \lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x = \lim_{x\to\infty} x\ln(1+\frac{7}{x}) = \lim_{x\to\infty} \frac{\ln(1+\frac{7}{x})}{\frac{1}{x}} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{7}{x}}\cdot\frac{-7}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{7}{x}}\cdot 7 = 7.$$

So $y = e^7 = \lim_{x\to\infty}(1+\frac{7}{x})^x$.

h) $\lim_{x\to0} \frac{\cos 4x - \cos 2x}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to0} \frac{-4\sin 4x + 2\sin 2x}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to0} \frac{-16\cos 4x + 4\cos 2x}{2} = -\frac{12}{2} = -6.$

i) $\lim_{x\to0} \frac{e^x-1-x}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to0} \frac{e^x-1}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to0} \frac{e^x}{2} = 2$

j) $\lim_{x\to0^+} 2x\ln x \overset{\text{l'Ho}}{=} \lim_{x\to0^+} \frac{2\ln x}{1/x} \overset{\text{l'Ho}}{=} \lim_{x\to0^+} \frac{2/x}{-1/x^2} = \lim_{x\to0^+} -\frac{2x^2}{x} = \lim_{x\to0^+} -2x = 0$

k) $\lim_{x\to\infty} x^2e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$

l) Use logs: $= \lim_{x\to\infty} \ln \frac{2x+9}{x+7}$. Use continuity: $= \ln(\lim_{x\to\infty} \frac{2x+9}{x+7}) \overset{\text{l'Ho}}{=} \ln(\frac{2}{1}) = \ln 2$

m) Assume $y = \lim_{x\to0^+}(2x)^x$. Then $\ln y = \lim_{x\to0^+} \ln(2x)^x = \lim_{x\to0^+} \frac{\ln(2x)}{1/x} \overset{\text{l'Ho}}{=} \lim_{x\to0^+} \frac{2/2x}{-1/x^2} = \lim_{x\to0^+} -\frac{x^2}{x} = \lim_{x\to0^+} -x = 0$. So $\ln y = 0$, therefore $y = e^0 = 1 = \lim_{x\to0^+}(2x)^x$.

n) Assume $y = \lim_{x\to0^+}(1+x)^{1/x}$. As above $\ln y = \lim_{x\to0^+} \ln(1+x)^{1/x} = \lim_{x\to0^+} \frac{\ln(1+x)}{x} \overset{\text{l'Ho}}{=} \lim_{x\to0^+} \frac{1/(1+x)}{1} = 1$. Therefore $y = e^1 = e = \lim_{x\to0^+}(1+x)^{1/x}$.

4. Find the limits of these sequences. Use the key limits when possible. When using l'Hopital's rule, switch to x.

a) $\displaystyle\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3$ (key limit)

b) $\displaystyle\lim_{n\to\infty} \ln(2n^2+7) - \ln(5n^2+n) = \lim_{x\to\infty} \ln\frac{2x^2+7}{5x^2+n} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \ln\frac{4x}{10x+1} \overset{\text{l'Ho}}{=} \ln\frac{4}{10}$

c) $\displaystyle\lim_{n\to\infty} \frac{2\ln(n+1)}{n^2} = \lim_{x\to\infty} \frac{2\ln(x+1)}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\frac{2}{x+1}}{2x} = 0$

d) $\displaystyle\lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$ (key limit)

e) $\lim_{n\to\infty} \left(\frac{-3}{2}\right)^n$ diverges (key limit)

f) $\lim_{n\to\infty} \frac{4n^2-3n+1}{5n^2+7} = \lim_{n\to\infty} \frac{4-\frac{3}{n}+\frac{1}{n^2}}{5+\frac{7}{n^2}} = \frac{4}{5}$

g) $\lim_{n\to\infty} \frac{n^2}{3^n} = \lim_{x\to\infty} \frac{x^2}{3^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2x}{3^x \ln 3} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2}{3^x(\ln 3)^2} = 0.$

5. Find the sums of these series, if they exist

a) $\sum_{n=0}^{\infty} 4\left(\frac{-2}{5}\right)^n = \frac{4}{1-(-\frac{2}{5})} = \frac{20}{7}$

b) $\sum_{n=0}^{\infty} 4\left(\frac{10}{9}\right)^n$ diverges since $|r| = \frac{10}{9} > 1$

c) $\sum_{n=1}^{\infty} \frac{2}{n^2+n} = \sum_{n=1}^{\infty} \frac{2}{n} - \frac{2}{n+1}$. $S_n = (2-\frac{2}{2}) + (\frac{2}{2}-\frac{2}{3}) + \cdots + (\frac{2}{n} - \frac{2}{n+1}) = 2 - \frac{2}{n+1}$.

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - \frac{2}{n+1} = 2$$

d) $\sum_{n=0}^{\infty} \frac{6}{n^2+7n+12} = \sum_{n=0}^{\infty} \frac{6}{n+3} - \frac{6}{n+4}$. $S_n = (\frac{6}{3}-\frac{6}{4}) + (\frac{6}{4}-\frac{6}{6}) + \cdots + (\frac{6}{n+3} - \frac{6}{n+4}) = 2 - \frac{6}{n+6}$.

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - \frac{6}{n+4} = 2$$

e) $\sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^1 - \left(\frac{1}{2}\right)^2 = \frac{1}{1-(\frac{1}{2})} - \frac{7}{4} = \frac{1}{4}$

f) $\sum_{n=1}^{\infty} 3\left(\frac{-2}{3}\right)^n = \sum_{n=0}^{\infty} 3\left(\frac{-2}{3}\right)^n - 3\left(\frac{-2}{3}\right)^0 = \frac{3}{1-(-\frac{2}{3})} - 3 = \frac{9}{5} - 3 = -\frac{6}{5}$

6. a) p-series with $p = 7/3 > 1$. The series series converges.

b) nth term test: $\lim_{n\to\infty} \frac{e^n}{1+e^n} = \lim_{x\to\infty} \frac{e^x}{1+e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \neq 0$. By the nth term test the series diverges. [Also integral test.]

c) Integral test. $f(x) = \frac{1}{16+9x^2}$ is continuous (rational) and is decreasing since the numerator is constant and the denominator increases as x does, so the entire function gets smaller.

$$\int_1^{\infty} \frac{1}{16+9x^2}\,dx = \lim_{b\to\infty} \frac{1}{12} \arctan 3x/4\Big|_1^b = \lim_{b\to\infty} \frac{1}{12}(\arctan 3b/4 - \arctan 3/4) = \frac{1}{12}(\pi/2 - \arctan 3/4).$$

Since the integral converges, so does the series.

d) nth term test: $\lim_{n\to\infty} \frac{2^n+1}{n^2} = \lim_{x\to\infty} \frac{2^x+1}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\ln 2 \cdot 2^x}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{(\ln 2)^2 2^x}{2} = \infty \neq 0$. By the nth term test the series diverges.

e) nth term test: $\lim_{n\to\infty} \ln(3n+3) - \ln(6n+2) = \lim_{n\to\infty} \ln \frac{(3n+3)}{(6n+2)} = \ln 12 \neq 0$. By the nth term test the series diverges.

f) nth term test: $\lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{-\frac{1}{x} \cdot \cos \frac{1}{x}}{-\frac{1}{x^2}} = \cos 0 = 1 \neq 0$. By the nth term test the series diverges.

g) Geometric series with $|r| = 3/11 < 1$. The series converges.

h) nth term test: $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{1}{1/x} = \infty \neq 0$. By the nth term test the series diverges.

i) Geometric series with $|r| = 5/4 \geq 1$. The series diverges.

j) Integral test. $f(x) = \frac{3x^2}{x^3+1}$ is continuous (rational) and $f'(x) = \frac{1-3x^4}{(x^3+1)^2} < 0$ for $x \geq 1$.

$$\int_1^\infty \frac{3x^2}{x^3+1}\,dx = \lim_{b\to\infty} \ln(x^3+1)\Big|_1^b = \lim_{b\to\infty} \ln(b^3+1) - \ln(x2) = \infty.$$

Since the integral diverges, so does the series.

k) nth term test: $\lim_{n\to\infty} \dfrac{(2n+1)!}{(2n-1)!} = \lim_{n\to\infty} (2n+1)(2n) = \infty \neq 0$. By the nth term test the series diverges.

l) Integral test. $f(x) = \frac{3}{x^2+5x+4}$ is continuous (rational) and is decreasing since the numerator is constant and the denominator increases as x does, so the entire function gets smaller.

$$\int_1^\infty \frac{3}{x^2+5x+4}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x+1} - \frac{1}{x+4}\,dx \lim_{b\to\infty} = \ln\left|\frac{x+1}{x+4}\right|\Big|_1^b = \lim_{b\to\infty} \ln\left|\frac{b+1}{b+4}\right| - \ln\left|\frac{2}{5}\right| = \ln(5/2).$$

Since the integral converges, so does the series.

m) nth term test: $\lim_{n\to\infty} \dfrac{n^n}{n!} = \infty \neq 0$ (Key Limit). By the nth term test the series diverges.

n) Integral test. $f(x) = \frac{2x}{e^{x^2}}$ is continuous (polynomial and exponential) and $f'(x) = \frac{2-4x^2}{e^{x^2}} < 0$ for $x \geq 1$.

$$\int_1^\infty \frac{2x}{e^{x^2}}\,dx = \lim_{b\to\infty} -\frac{1}{e^{x^2}}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{e^{b^2}} + \frac{1}{e} = \frac{1}{e}.$$

Since the integral converges, so does the series.

o) Integral test. $f(x) = \frac{2}{x^2+4x+3}$ is continuous (rational) and is decreasing since the numerator is constant and the denominator increases as x does, so the entire function gets smaller.

$$\int_1^\infty \frac{2}{x^2+4x+3}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x+1} - \frac{1}{x+3}\,dx \lim_{b\to\infty} = \ln\left|\frac{x+1}{x+3}\right|\Big|_1^b = \lim_{b\to\infty} \ln\left|\frac{b+1}{b+3}\right| - \ln\left|\frac{2}{4}\right| = \ln 2.$$

Since the integral converges, so does the series.

p) p-series test: $p = 3 > 1$ so the series converges. (Also integral test.)