Sequences

Our ultimate goal by the end of the course is to approximate functions by using polynomials with an infinite number of terms. Such approximations are what allows your calculator to evaluate log, trig, and exponential functions (which are types of *transcendental functions*[1]). Similar approximations are used in data compression so that your iPod can play music files that are relatively small. We want to use polynomials rather than trig functions, logs, or exponentials because they are easier to work with. The down side is that this forces us to deal with infinity, or more precisely, limits of "infinite sums which turn out to be very interesting!

[1] Roughly speaking, there are two types of functions: *algebraic* and *transcendental*. The algebraic functions are created by using the elementary operations of addition, subtraction, multiplication, division, and root extraction. Transcendental functions are all those functions that are *not* algebraic. In other words, a function which "transcends," i.e., cannot be expressed in terms of, algebra.

Introduction

Our first topic is sequences. You have probably seen sequences on SAT or IQ tests where you had to figure out the next term or the general pattern. Here are several examples.

EXAMPLE 12.0.1. See if you can figure out the next term and the general formula for the nth term for each sequence or list of numbers.

a. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
b. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
c. $0, -1, 4, -9, 16, \ldots$
d. $0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}, \ldots$
e. $1, 2, 6, 24, 120, \ldots$

DEFINITION 12.1. A **sequence** of real numbers is a function $f(n)$ whose domain is the set of all positive integers n. Notation: Instead of using $f(n)$ we usually use a_n and indicate the entire sequence by $\{a_n\}_{n=1}^{\infty}$ or just $\{a_n\}$. More generally, a sequence can start with any integer m in which case the domain consists of all integers $n \geq m$. ($m = 0$ is a common starting value.)

EXAMPLE 12.0.2 (Continued). Here are the functions (formulæ) describing the sequences in Example 12.0.1.

a. $f(n) = \frac{1}{2^n}$, $n \geq 0$; $\left\{\frac{1}{2^n}\right\}_{n=0}^{\infty}$.
b. $a_n = \frac{1}{n}$, $n \geq 1$; $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.
c. $\{(-1)^n n^2\}_{n=0}^{\infty}$.
d. $\left\{1 + \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$.
e. $\{n!\}_{n=1}^{\infty}$.

Sequences can be described in two ways. Often we will use an **explicit formula** like we would with an ordinary function). For instance

EXAMPLE 12.0.3. The sequence $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$ can be described in several ways. For example, we might use $a_n = f(n) = \frac{n-1}{n}$ for $n \geq 1$, so the sequence would be $\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$. Here we can compute a_n explicitly from the given formula.

Such a formula is not unique. We might start with $n = 0$ and use $b_n = g(n) = \frac{n}{n+1}$ for $n \geq 0$, so the sequence would be $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$.

Another way that sequences are defined is with a **recurrence relation**. We specify the first term of the sequence and give a general rule for computing the next term of the sequence from the previous ones. For example,

$$a_1 = 1,\ a_{n+1} = a_n^2 - 1.$$

The first few terms are

$$1, 0, -1, 0, -1, 0, -1, \ldots$$

Another such sequences is the **factorial** sequence (function). . The function n-factorial is denoted by $n!$. We define

$$a_0 = 0! = 1, \; a_{n+1} = (n+1)! = n! \cdot (n+1)$$

So the first few terms are

$$\begin{aligned} 0! &= 1 \\ 1! &= 0! \cdot 1 = 1 \cdot 1 = 1 \\ 2! &= 1! \cdot 2 = 1 \cdot 2 = 2 \\ 3! &= 2! \cdot 3 = 1 \cdot 2 \cdot 3 = 6 \\ 4! &= 3! \cdot 4 = 1 \cdot 2 \cdot 3 \cdot 4 = 24 \end{aligned}$$

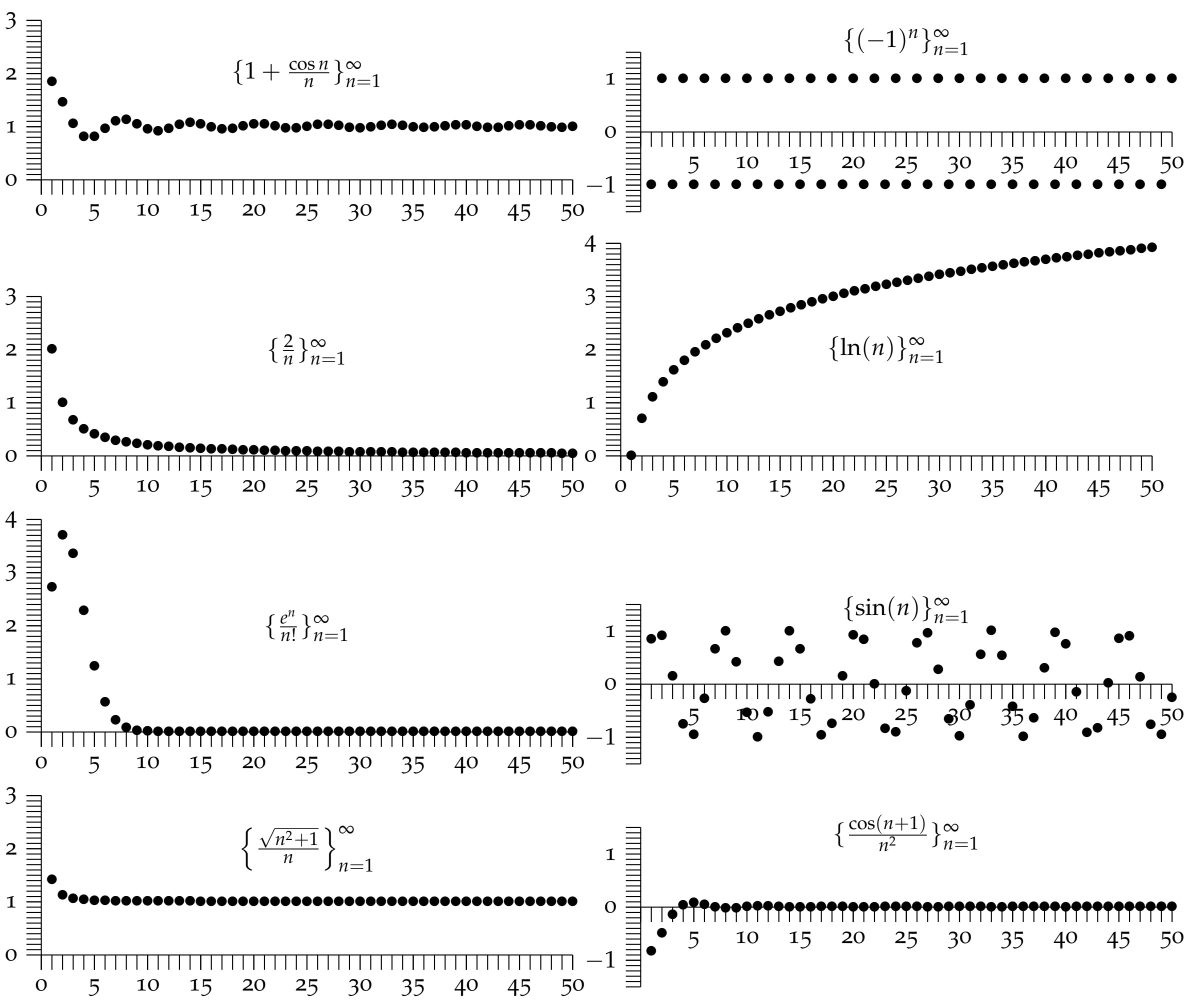
You can see that we can also give an explicit formula for $n!$:

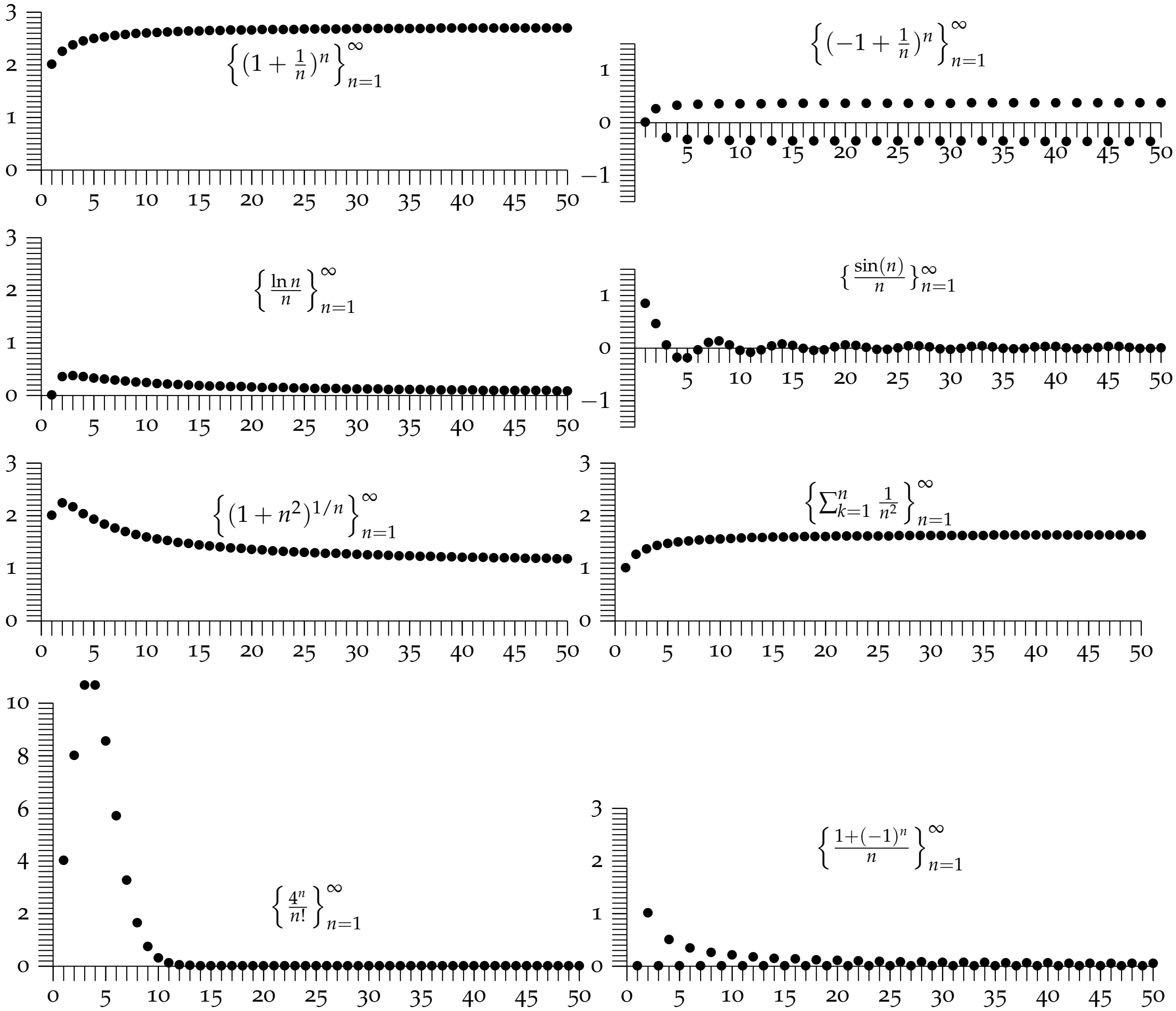
$$n! = 1 \cdot 2 \cdots n$$

For example, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

YOU TRY IT 12.1. The factorial function gets large very fast. Try some values on your calculator. What is the smallest value of n so that $n! > 1,000,000$?

Though we don't often do it, we can graph sequences. Here are some examples.





Caution: One important thing to notice: These sequences (functions) are *not continuous* because they are not defined on intervals. Their graphs are just an infinite series of dots, one for each integer in the domain of the sequence.

Limits of Sequences

In some of the graphs above, as n gets large, the terms in the sequence seem to be approaching a particular value. For example the next to last sequence $\left\{\frac{4^n}{n!}\right\}_{n=1}^{\infty}$ appears to approach 0 as n gets large. On the other hand, $\{\sin n\}_{n=1}^{\infty}$ does not appear to approach any particular value as n gets large. We can adapt the language of limits to this situation.

DEFINITION 12.2 (Informal)**.** A sequence $\{a_n\}_{n=1}^{\infty}$ has a **limit** L if we can make a_n arbitrarily close to L by taking n sufficiently large. We denote this by writing

$$\lim_{n\to\infty} a_n = L.$$

EXAMPLE 12.0.4. Here's a familiar sequence from 'back in the day' when we were working with Riemann sums.

(*a*) Let

$$\{u_n\}_{n=1}^{\infty} = \left\{\frac{n(n+1)}{2n^2}\right\}_{n=1}^{\infty}.$$

We evaluate this limit easily by using a bit of algebra (or use l'Hôpital's rule):

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1+0}{2} = \frac{1}{2}.$$

Another way to solve algebraic limits at infinity like this one is to focus on the highest powers in algebraic or rational functions.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{n^2+n}{n^2} \stackrel{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n^2}{2n^2} = \frac{1}{2}.$$

Still another way is to use l'Hôpital's rule, but see Theorem 12.0.5.

(*b*) Another similar example we saw was

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{n(n+1)(2n+1)}{6n^3}\right\}_{n=1}^{\infty}.$$

Since this is an algebraic limit at infinity, let's use highest powers:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$
$$= \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} \stackrel{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n^2}{6n^2} = \frac{1}{3}.$$

(*c*) Here's a 'backwards' sequence as an illustration of how general the sequence concept is: The index goes backwards to $-\infty$. Let

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{\sqrt{5n^2+n+1}}{2n+3}\right\}_{n=-1}^{-\infty}.$$

Since this is a limit at infinity, we can focus on the 'highest powers' to simplify the limit:

$$\lim_{n\to-\infty} a_n = \lim_{n\to-\infty} \frac{\sqrt{5n^2+n+1}}{2n+3} \stackrel{\text{HPwrs}}{=} \lim_{n\to-\infty} \frac{\sqrt{5n^2}}{2n}.$$

Now here is where you need to be very careful: Remember that $\sqrt{x^2} = |x|$, NOT x. (Try a few negative values of x to see why.) Further, when x is negative, $|x| = -x$. So continuing the calculation above, since the n-values are negative

$$\lim_{n\to-\infty} a_n = \lim_{n\to-\infty} \frac{\sqrt{5n^2}}{2n} = \lim_{n\to-\infty} \frac{|n|\sqrt{5}}{2n} = \lim_{n\to-\infty} \frac{-n\sqrt{5}}{2n} = -\frac{\sqrt{5}}{2}.$$

YOU TRY IT 12.2. Return to the graphs of the sequences given earlier. By inspecting the graphs (no calculations), which appear to have limits and which do not?

The following do not have limits: $\{(-1)^n\}_{n=1}^{\infty}$, $\{\ln n\}_{n=1}^{\infty}$, $\{\sin n\}_{n=1}^{\infty}$, $\left\{\left(-1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$.

Some sequence limits are more challenging. Consider the sequence $\{a_n\}_{n=1}^{\infty} = \left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$. Does it have a limit? We might try to evaluate

$$\lim_{n\to\infty} \frac{\ln n}{n}$$

by using l'Hôpital's rule. However, to apply l'Hôpital's rule the numerator and denominator of the sequence need to be differentiable. But in a sequence, these functions are not even continuous, let alone differentiable. Fortunately there is a way around this.

THEOREM 12.0.5 (Key Fact). Suppose that f is a function so that $\lim_{x\to\infty} f(x) = L$. If a_n is a sequence such that $f(n) = a_n$ for all integers n in the domain of the sequence, then $\lim_{n\to\infty} a_n = L$, too.

Essentially this says that if we can 'convert' a sequence to a corresponding function of x and evaluate the resulting limit, then the sequence has the same limit as the function.

EXAMPLE 12.0.6. Return to the sequence $\{a_n\}_{n=1}^{\infty} = \left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$. The sequence can be described by the formula $a_n = f(n) = \frac{\ln n}{n}$. So if we let $f(x) = \frac{\ln x}{x}$ (for $x > 0$), then we can try to evaluate the limit as $x \to \infty$ using any method from Calculus I. In particular we can apply l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0.$$

Therefore by the Key Fact Theorem 12.0.5 above,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = 0,$$

also!

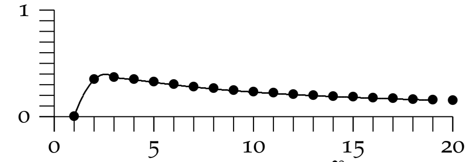


Figure 12.1: The sequence $\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$ and the function $f(x) = \frac{\ln x}{x}$ that 'connects the dots.' Both have the same limit at infinity.

YOU TRY IT 12.3. Here are several that you should try now. Many of these will look familiar from our recent work with l'Hôpital's rule.

(*a*) $\lim_{n\to\infty} \frac{2n}{n+1}$.

(*b*) $\lim_{n\to\infty} \frac{1-n^2}{6n+7}$.

(*c*) $\lim_{n\to\infty} \frac{e^n}{n^2+1}$.

(*d*) $\lim_{n\to\infty} (1+2n)^{1/n}$.

(*e*) $\lim_{n\to\infty} \frac{\ln \sqrt{n}}{n}$.

(*f*) $\lim_{n\to\infty} (\sqrt{n})^{1/n}$.

(*g*) $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n/2}$.

(*h*) $\lim_{n\to\infty} (\ln(3n+5) - \ln(4n+6))$.

Answers: (a) 2; (b) DNE $(-\infty)$; (c) DNE $(+\infty)$; (d) 1; (e) 0; (f) 1; (g) $e^{-1/2}$; (h) $\ln(3/4)$.