Important Limits

The sequence $\left\{\left(1+\frac{k}{n}\right)^n\right\}_{n=1}^{\infty}$

Finding the limit of the sequence $\left\{\left(1+\frac{k}{n}\right)^n\right\}_{n=1}^{\infty}$ should remind you of some earlier work we did with l'Hôpital's rule. Notice that $\lim_{n\to\infty}\left(1+\dfrac{k}{n}\right)^n$ has the indeterminate form 1^∞. To determine limits of this form, it is useful to use the natural log function.

Let $y=\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n$. Taking the logs of both sides and switching the order of the limit and the log (the log function is continuous) we get:

$$\ln y=\ln\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n=\lim_{n\to\infty}\ln\left(1+\frac{k}{n}\right)^n=\lim_{n\to\infty}n\ln\left(1+\frac{k}{n}\right).$$

To evaluate the limit we switch to the continuous variable x and put the limit in $\frac{0}{0}$ indeterminate form so that we can eventually use l'Hôpital's rule.

$$\ln y=\lim_{x\to\infty}x\ln\left(1+\frac{k}{x}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{k}{x}\right)}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\cdot\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{k}{1+\frac{k}{x}}=\frac{k}{1+0}=k.$$

Since $\ln y=k$, then $y=e^k$ which means that $\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n=e^k$.

The sequence $\left\{n^{1/n}\right\}_{n=1}^{\infty}$

This time $\lim_{n\to\infty}n^{1/n}$ or $\lim_{n\to\infty}\sqrt[n]{n}$ has the indeterminate form ∞^0. We use the same method as in the previous situation. Let $y=\lim_{n\to\infty}n^{1/n}$. Then

$$\ln y=\ln\lim_{n\to\infty}n^{1/n}=\lim_{n\to\infty}\ln n^{1/n}=\lim_{n\to\infty}\frac{\ln n}{n}.$$

To evaluate the limit we switch the the continuous variable x and use l'Hôpital's rule since the limit is now in the indeterminate form $\frac{\infty}{\infty}$.

$$\ln y=\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0.$$

Since $\ln y=0$, then $y=e^0=1$ which means that $\lim_{n\to\infty}n^{1/n}=\lim_{n\to\infty}\sqrt[n]{n}=1$.

The sequence $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$

This sequence is a little simpler to deal with: Notice that

$$0\le\frac{n!}{n^n}=\frac{1\cdot 2\cdot 3\cdots(n-1)\cdot n}{n\cdot n\cdot n\cdots n\cdot n}=\frac{1}{n}\cdot\frac{2\cdot 3\cdots(n-1)\cdot n}{n\cdot n\cdots n\cdot n}\le\frac{1}{n}\cdot 1.$$

This means that

$$0\le\frac{n!}{n^n}\le\frac{1}{n}.$$

So taking limits and applying the squeeze theorem we obtain

$$\lim_{n\to\infty}0\le\lim_{n\to\infty}\frac{n!}{n^n}\le\lim_{n\to\infty}\frac{1}{n}\Rightarrow 0\le\lim_{n\to\infty}\frac{n!}{n^n}\le 0.$$

Therefore, $\lim_{n\to\infty}\frac{n!}{n^n}=0$.

YOU TRY IT 12.4. Give an argument that shows $\lim_{n\to\infty}\dfrac{n^n}{n!}=\infty$ (diverges). Hint: Show that $\frac{n^n}{n!}>n$.

Sequences of the form $\{r^n\}_{n=1}^{\infty}$

For each fixed real number r, we can form the sequence $\{r^n\}_{n=1}^{\infty}$. For some values of r, the sequence converges and for others it does not.

EXAMPLE 12.0.7. By inspection we see that

(a) $\left\{\left(\frac{1}{2}\right)^n\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\}$ and so $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$.

(b) $\left\{\left(-\frac{1}{4}\right)^n\right\}_{n=1}^{\infty} = \left\{-\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \dots\right\}$ and so $\lim_{n\to\infty} \left(-\frac{1}{4}\right)^n = 0$.

(c) $\{2^n\}_{n=1}^{\infty} = \{2, 4, 8, 16, \dots\}$ and so $\lim_{n\to\infty} (2)^n$ diverges to $+\infty$.

(d) $\{(-3)^n\}_{n=1}^{\infty} = \{-3, 9, -27, 81, \dots\}$ and so $\lim_{n\to\infty} (-3)^n$ diverges.

(e) $\{1^n\}_{n=1}^{\infty} = \{1, 1, 1, 1, \dots\}$ and so $\lim_{n\to\infty} (1)^n = 1$.

(f) $\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \dots\}$ and so $\lim_{n\to\infty} (-1)^n$ diverges.

We see that if $|r| < 1$ then the powers of r get small and converge to 0. If $|r| > 1$, then the powers of r get large (without bound) in magnitude and so the sequence diverges. This is summarized below.

Summary of Key Limits

You should know and be able to use all of the following limits.

THEOREM 12.0.8. Summary of important limits.

(a) $\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = e^k$. In particular $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

(b) $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \sqrt[n]{n} = 1$.

(c) $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ and $\lim_{n\to\infty} \frac{n^n}{n!} = \infty$ (diverges).

(d) Consider the sequence $\{r^n\}_{n=1}^{\infty}$, where r is a real number.

1. If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$;
2. If $r = 1$, then $\lim_{n\to\infty} r^n = 1$;
3. Otherwise $\lim_{n\to\infty} r^n$ does not exist (diverges).

A final note about sequences. It should be clear that

$$\text{if } \lim_{n\to\infty} a_n = L \text{ then } \lim_{n\to\infty} a_{n+1} = L \text{ and } \lim_{n\to\infty} a_{n-1} = L \tag{12.1}$$

since the terms in the infinite tails of the sequences are the same.

12.1 Problems

1. (a) List the first four terms of the sequence $\left\{\frac{n+1}{3n-1}\right\}_{n=1}^{\infty}$.

(b) Find a formula for a_n for the sequence $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\}$.

(c) Find a formula for a_n for the sequence $\{\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \dots\}$.

2. Determine whether the sequence converges or diverges. If it converges, find the limit. Use l'Hôpital's rule where appropriate. Use log properties for (d).

(a) $\left\{\frac{\sqrt{n}}{1+n}\right\}_{n=1}^{\infty}$ (b) $\left\{\left(1 + \frac{2}{n}\right)^n\right\}_{n=1}^{\infty}$ (c) $\left\{\frac{\ln n^2}{n}\right\}_{n=1}^{\infty}$ (d) $\{\ln(2n+1) - \ln(3n)\}_{n=1}^{\infty}$

3. Find these limit (if they exists):

(a) $\{a_n\}_{n=1}^{\infty} = \left\{ \frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{n}{n^2} \right\}_{n=1}^{\infty}$ (b) $\{a_n\}_{n=2}^{\infty} = \left\{ \int_1^{\infty} \frac{1}{x^n}\, dx \right\}_{n=2}^{\infty}$

4. Find the limits of these sequences. Use the key limits when possible.

(a) $\left\{ \left(1 + \frac{3}{n}\right)^n \right\}_{n=1}^{\infty}$ (b) $\left\{ \ln(2n^2+7) - \ln(5n^2+n) \right\}_{n=1}^{\infty}$ (c) $\left\{ \frac{2\ln(n+1)}{n^2} \right\}_{n=1}^{\infty}$

(d) $\left\{ \left(\frac{2}{3}\right)^n \right\}_{n=1}^{\infty}$ (e) $\left\{ \left(\frac{-3}{2}\right)^n \right\}_{n=1}^{\infty}$ (f) $\left\{ \frac{4n^2-3n+1}{5n^2+7} \right\}_{n=1}^{\infty}$

5. Find the limits of the following sequences.

(a) $\left\{ \frac{3n}{n+1} \right\}_{n=1}^{\infty}$ (b) $\left\{ \frac{1-n^2}{6n+7} \right\}_{n=1}^{\infty}$ (c) $\left\{ \frac{e^n}{n^2+1} \right\}_{n=1}^{\infty}$

(d) $\left\{ (1+2n)^{\frac{1}{n}} \right\}_{n=1}^{\infty}$ (e) $\left\{ \frac{\ln\sqrt{n}}{n} \right\}_{n=1}^{\infty}$ (f) $\left\{ (\sqrt{n})^{\frac{1}{n}} \right\}_{n=1}^{\infty}$ (g) $\left\{ \left(1 - \frac{1}{n}\right)^{\frac{n}{2}} \right\}_{n=1}^{\infty}$

12.2 *Terminology for Sequences*

There are a few more basic terms that will be used to describe sequences, terms which are similar to those used for more general functions.

DEFINITION 12.3. A sequence $\{a_n\}_{n=1}^{\infty}$ is **nondecreasing** if each term is at least a big as its predecessor: $a_{n+1} \geq a_n$ for all n. Similarly it is **nonincreasing** if $a_{n+1} \leq a_n$ for all n. A sequence that is either nondecreasing or nonincreasing is said to be **monotonic**.

Two simple examples illustrate the idea:

$$\{a_n\}_{n=1}^{\infty} = \left\{ 1 + \frac{1}{n} \right\}_{n=1}^{\infty} = \{2, 3/2, 4/3, 5/4, \dots\}$$

is nonincreasing while

$$\{b_n\}_{n=1}^{\infty} = \left\{ 1 - \frac{1}{n} \right\}_{n=1}^{\infty} = \{0, 1/2, 2/3, 3/4, 4/5, \dots\}$$

is nondecreasing. The sequence

$$\{c_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, -1, \dots\}$$

is neither nonincreasing nor nondecreasing, so it is not monotonic.

YOU TRY IT 12.5. Look back to the sequences that were plotted beginning of this section and pick out the the nonincreasing and the nondecreasing ones and those that were not monotonic.

DEFINITION 12.4. A sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded** if there is some number M so that $|a_n| \leq M$ for all n.

The three sequences above are bounded. For

$$\{a_n\}_{n=1}^{\infty} = \left\{ 1 + \frac{1}{n} \right\}_{n=1}^{\infty} = \{2, 3/2, 4/3, 5/4, \dots\}$$

notice $|a_n| \leq 2$. For

$$\{b_n\}_{n=1}^{\infty} = \left\{ 1 - \frac{1}{n} \right\}_{n=1}^{\infty} = \{0, 1/2, 2/3, 3/4, 4/5, \dots\}$$

notice $|b_n| \leq 1$ and for

$$\{c_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, -1, \dots\}$$

we have $|c_n| \leq 1$. Observe that we could have used larger bounds for each, e.g., $|a_n| \leq 12$.

From our perspective, the most important fact is that

THEOREM 12.2.1. If a sequence $\{a_n\}_{n=1}^{\infty}$ is both monotone and bounded, then it converges.

This is a hard theorem to prove and requires a more advanced understanding of the real numbers. Take Math 331.

As an example, the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ above were both monotone and bounded and both converge:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 + \frac{1}{n} = 1$$

while

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1 - \frac{1}{n} = 1.$$