Sigma notation

2.1 Introduction

We use **sigma notation** to indicate the summation process when we have several (or infinitely many) terms to add up. You may have seen sigma notation in earlier courses. It is used to indicate the summation of a number of terms that follow some pattern.

This tells us to end with $i = n$
$\downarrow$

$$\Sigma \text{ tells us to sum} \rightarrow \sum_{i=k}^{n} a_i \leftarrow$$

a_i tells us what to sum. Usually a_i is a function of i.

$\uparrow$
This tells us to start with $i = k$

Figure 2.1: The schematics of sigma notation.

EXAMPLE 2.1. Here are several simple examples. Note the the letter for the index need not be n.

(a) $\displaystyle\sum_{1=1}^{5} 2i = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) = 30.$

(b) $\displaystyle\sum_{i=3}^{7} (i^2 + 1) = 10 + 17 + 26 + 37 + 50 = 140.$

(c) $\displaystyle\sum_{j=1}^{4} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$

(d) In this example n represents some fixed but unknown integer value. Notice that k changes but n does not.

$$\sum_{k=1}^{n} \frac{1}{n}(2k+3) = \frac{1}{n}(5) + \frac{1}{n}(7) + \cdots + \frac{1}{n}(2n+3).$$

(e) Note that the entire summation may be symbolic:

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

(f) Here's another symbol-laden summation:

$$\sum_{i=1}^{n} f\left(2 + \tfrac{3i}{n}\right)\tfrac{3}{n} = f\left(2 + \tfrac{3}{n}\right)\tfrac{3}{n} + f\left(2 + \tfrac{6}{n}\right)\tfrac{3}{n} + f\left(2 + \tfrac{9}{n}\right)\tfrac{3}{n} + \cdots f\left(2 + \tfrac{3n}{n}\right)\tfrac{3}{n}.$$

If $f(x) = x^2$, then the previous sum would become

$$\sum_{i=1}^{n} \left(2 + \tfrac{3i}{n}\right)^2 \tfrac{3}{n} = \left(2 + \tfrac{3}{n}\right)^2 \tfrac{3}{n} + \left(2 + \tfrac{6}{n}\right)^2 \tfrac{3}{n} + \left(2 + \tfrac{9}{n}\right)^2 \tfrac{3}{n} + \cdots (5)^2\,\tfrac{3}{n}.$$

(g) You should be able to reverse this process and write a sum in compact summation notation by recognizing appropriate patterns.

$$2+4+6+8+\cdots+20=\sum_{k=1}^{10} 2k.$$

Or you might think

$$2+4+6+8+\cdots+20=2(1+2+3+4+\cdots+10)=2\sum_{k=1}^{10} k.$$

(h) This reverse process might take place symbolically:

$$f(1.2)(0.2)+f(1.4)(0.2)+\cdots+f(2.0)(0.2)=\sum_{i=1}^{5} f(1+0.2i)(0.2)$$

or as

$$(0.2)[f(1.2)+f(1.4)+\cdots+f(2.0)]=0.2\sum_{i=1}^{5} f(1+0.2i)$$

YOU TRY IT 2.1 (Sigma Notation). Translate each of the following:

(a) $\sum_{n=1}^{4} n^2$ (b) $\sum_{n=2}^{5} \cos(n\pi)$ (c) $\sum_{n=0}^{4} n^3+2n$

Now write each of the following sums using sigma notation.

(d) $6+9+12+15+18$ (e) $3+9+27+81+243$

(f) $-1+1-1+1-1+1$ (g) $0+1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{20}$

Answers to YOU TRY IT 2.1 :

(a) $1+4+9+16=30$

(b) $1-1+1-1=0$

(c) $0+3+12+33+72=120$

(d) $3\sum_{n=2}^{6} n$ (e) $\sum_{n=1}^{5} 3^n$

(f) $\sum_{n=1}^{6} (-1)^n$ (g) $\sum_{n=0}^{20} \sqrt{n}$

YOU TRY IT 2.2 (Challenge). Determine the values for each of these sums for $n=5$, 10, and 20.

(a) $\sum_{k=1}^{n} \frac{1}{k}$ (b) $\sum_{k=1}^{n} \frac{1}{k^2}$ (c) $\sum_{k=0}^{n} \frac{1}{2^k}$

(d) Now determine a general formula for any value of n for the sum $\sum_{k=0}^{n} \frac{1}{2^k}$.

Notice in some of the sums we factored out a constant term. In fact applying basic associativity and distributivity laws for addition, we have

$$\sum_{i=1}^{n} ca_i = ca_1+ca_2+\cdots+ca_n = c(a_1+a_2+\cdots+a_n)=c\sum_{i=1}^{n} a_i$$

and

$$\sum_{i=1}^{n}(a_i+b_i)=(a_1+b_2)+(a_2+b_2)+\cdots+(a_n+b_n)$$

$$=(a_1+a_2+\cdots+a_n)+(b_1+b_2+\cdots+b_n)=\sum_{i=1}^{n} a_i+\sum_{i=1}^{n} b_i.$$

In other words, we have

THEOREM 2.1 (Basic Summation properties). For any constant c,

(a) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

(b) $\sum_{i=1}^{n}(a_i+b_i)=\sum_{i=1}^{n} a_i+\sum_{i=1}^{n} b_i.$

2.2 *Four Key Summation Formulæ*

As we begin to tackle the area problem we will find that we need to use a few basic summation formulæ repeatedly.

The first is quite easy to see. Suppose that c is a constant. Then

$$\sum_{i=1}^{n} c = \overbrace{c + c + \cdots + c}^{n \text{ times}} = nc.$$

Another simple formula that you can figure out is the sum of the first n integers. Let $S_n = 1 + 2 + 3 + \cdots + (n-1) + n$. For example, $S_4 = 1 + 2 + 3 + 4 = 10$ (as any bowler would know). There a formula for S_n that C. F. Gauss (a very famous 19th century mathematician) figured out when he was 6. Here's how: Write the summands forwards and backwards like so:

$$\begin{array}{rrcccccccccc} & S_n = & 1 & + & 2 & + & 3 & + \cdots + & (n-1) & + & n \\ + & S_n = & n & + & (n-1) & + & (n-2) & + \cdots + & 2 & + & 1 \\ \hline & 2S_n = & & & & & & & & & \end{array}$$

1. Now add each column. What do you get as the total for each? How many times?

2. So what is the formula for $2S_n$? Now solve for S_n.

3. Use your formula to check that $S_4 = 10$. Now use it to determine S_{10} and S_{100}.

4. Suppose you wanted to sum the even integers only. Let T_n be the sum of the first n even integers: $T_n = 2 + 4 + \cdots + 2n$. What is the formula for T_n?

Ok, your formula for the sum of the first n integers should have been $S_n = \frac{n(n+1)}{2}$. There are two more formulæ that you should memorize: the sum of the first n integers *squared* and *cubed*. They are listed below. Their proofs are a little harder and we will skip them here.

THEOREM 2.2 (Summation Formulæ). For any positive integer n

(*a*) $\displaystyle\sum_{i=1}^{n} c = nc$

(*b*) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(*c*) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(*d*) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Interesting! The sum the first n cubes is the square of the sum of the first n integers.

EXAMPLE 2.2. Let's use these formulæ to calculate a few sums.

(*a*) $\displaystyle\sum_{i=1}^{8} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{8(9)(17)}{6} = 204.$

(*b*) This next problem is more typical: We have an unknown upper limit for the sum. The goal is to express the answer as simply as possible.

$$\begin{aligned}\sum_{i=1}^{n} \frac{3i^2}{n^3} = \frac{3}{n^3}\sum_{i=1}^{n} i^2 = \frac{3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] &= \frac{(n+1)(2n+1)}{2n^2} \\ &= \frac{2n^2+3n+1}{2n^2} \\ &= \frac{2n^2+3n+1}{2n^2} \\ &= 1 + \frac{3}{2n} + \frac{1}{2n^2}.\end{aligned}$$

(*c*) Now use limits to determine the value of the previous sum as $n \to \infty$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3i^2}{n^3} = \lim_{n\to\infty} 1 + \frac{3}{2n} + \frac{1}{2n^2} = 1 + 0 + 0 = 1.$$

(*d*) The ability to determine limits like the one above will be critical for solving the area problem. Here's another. Determine

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i-3}{n^2}.$$

First we work on the sum using Theorem 2.1 and 2.2

$$\begin{aligned}\sum_{i=1}^{n} \frac{2i-3}{n^2} = \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{3}{n^2}\sum_{i=1}^{n} 1 = \frac{2}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{3}{n^2}(n) &= \frac{n+1}{n} - \frac{3}{n} \\ &= \frac{n-2}{n} \\ &= 1 - \frac{2}{n}.\end{aligned}$$

So

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i-3}{n^2} = \lim_{n\to\infty} 1 - \frac{2}{n} = 1 - 0 = 1.$$

(*e*) Let $S_n = \sum_{i=1}^{n} \frac{4}{n}\left[\left(\frac{i}{n}\right)^3 + 3\right]$. Determine $\lim_{n\to\infty} S_n$.

SOLUTION. This time

$$\begin{aligned}S_n = \sum_{i=1}^{n} \frac{4}{n}\left[\left(\frac{i}{n}\right)^3 + 3\right] = \frac{4}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3 + \frac{4}{n}\sum_{i=1}^{n} 3 = \frac{4}{n^4}\sum_{i=1}^{n} i^3 + \frac{4}{n}(3n) &= \frac{4}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] + 12 \\ &= \frac{(n+1)^2}{n^2} + 12 \\ &= \frac{n^2+2n+1}{n^2} + 12 \\ &= 1 + \frac{2}{n} + \frac{1}{n^2} + 12 \\ &= 13 + \frac{2}{n} + \frac{1}{n^2}.\end{aligned}$$

So

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 13 + \frac{2}{n} + \frac{1}{n^2} = 13.$$

YOU TRY IT 2.3. Use summation properties and formulæ to find the following general sums. Your answer will be in terms of n. Be sure to **simplify**.

$$(a)\ \sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \qquad (b)\ \sum_{i=1}^{n}\frac{i^2-1}{n^3} \qquad (c)\ \sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^2\left(\frac{1}{n}\right)$$

Now use your answers to (a) and (b) to determine

$$(d)\ \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$(e)\ \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^2-1}{n^3}$$

Answers to YOU TRY IT 2.3 :

$$(c)\ \frac{7}{3}+\frac{3}{2n}+\frac{1}{6n^2} \qquad (d)\ 2 \qquad (e)\ \frac{1}{3}$$

YOU TRY IT 2.4. Here are two more to practice.

(a) Let $S_n = \sum_{i=1}^{n}\frac{2}{n}\left[2-2\left(\frac{i}{n}\right)^2\right]$. Determine $\lim_{n\to\infty} S_n$.

(b) Let $R_n = \sum_{i=1}^{n}\frac{12}{n}\left[\left(\frac{i}{n}\right)^2-\frac{2i}{n}\right]$. Determine $\lim_{n\to\infty} R_n$.

Answers to YOU TRY IT 2.4 :

$$(a)\ \frac{8}{3} \qquad (b)\ -8$$

A Look Ahead

This next example is much more typical of how we will actually use summation notation.

EXAMPLE 2.2.1. Let $f(x) = 2-\frac{1}{2}x^2$ on the closed interval $[0,2]$. Determine the value of

$$\sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n},$$

where n is an arbitrary positive integer.

SOLUTION. First we must evaluate $f\left(\frac{2i}{n}\right)$.

$$f\left(\frac{2i}{n}\right) = 2-\frac{1}{2}\left(\frac{2i}{n}\right)^2 = 2-\frac{1}{2}\left(\frac{4i^2}{n^2}\right) = 2-\frac{2i^2}{n^2}. \tag{2.1}$$

Now turning to the sum

$$\begin{aligned}
\sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n} &= \sum_{i=1}^{n}\left[2-\frac{2i^2}{n^2}\right]\frac{2}{n} && \text{Using (2.1)}\\
&= \frac{2}{n}\sum_{i=1}^{n}\left[2-\frac{2i^2}{n^2}\right] && n \text{ is constant, use Theorem 2.1(a)}\\
&= \frac{2}{n}\sum_{i=1}^{n}2-\frac{2}{n}\sum_{i=1}^{n}\frac{2i^2}{n^2} && \text{Split the sum, use Theorem 2.1(b)}\\
&= \frac{2}{n}\cdot 2n-\frac{4}{n^3}\sum_{i=1}^{n}i^2 && \text{Theorem 2.2(a) and Theorem 2.1(a)}\\
&= 4-\frac{4}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} && \text{Theorem 2.2(c)}\\
&= 4-\frac{2(n+1)(2n+1)}{3n^2} && \text{Simplify}\\
&= 4-\frac{4n^2+6n+2}{3n^2} && \text{Simplify}\\
&= 4-\frac{4}{3}-\frac{2}{n}-\frac{2}{3n^2} && \text{Simplify and distribute the minus sign}\\
&= \frac{8}{3}-\frac{2}{n}-\frac{2}{3n^2}.
\end{aligned}$$

Notice that we can use this general solution to evaluate sums for various values of n without having to redo the sum. For example if $n = 10$, then

$$\sum_{i=1}^{10} f\left(\frac{2i}{10}\right) \cdot \frac{2}{10} = \frac{8}{3} - \frac{2}{10} - \frac{2}{3(10)^2} = 2.46.$$

More generally, as $n \to \infty$, it is easy to see that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right) \cdot \frac{2}{n} = \lim_{n\to\infty} \left(\frac{8}{3} - \frac{2}{n} - \frac{2}{3n^2}\right) = \frac{8}{3}. \tag{2.2}$$

What do sums such as those in Example 2.2.1 represent? Why should we be interested in them? Well, $f\left(\frac{2i}{n}\right)$ represents the of a function at n equally spaced points. The points are $\frac{2}{n}$ units apart. We can think of the product $f\left(\frac{2i}{n}\right) \cdot \frac{2}{n}$ as "height $\times$ width" or the area of a rectangle. Thus the sum represents the area of rectangles that approximate the area under the curve of f. As the figures below indicate, as the number of rectangles n gets larger, we get a more precise approximation to the actual area under the curve.

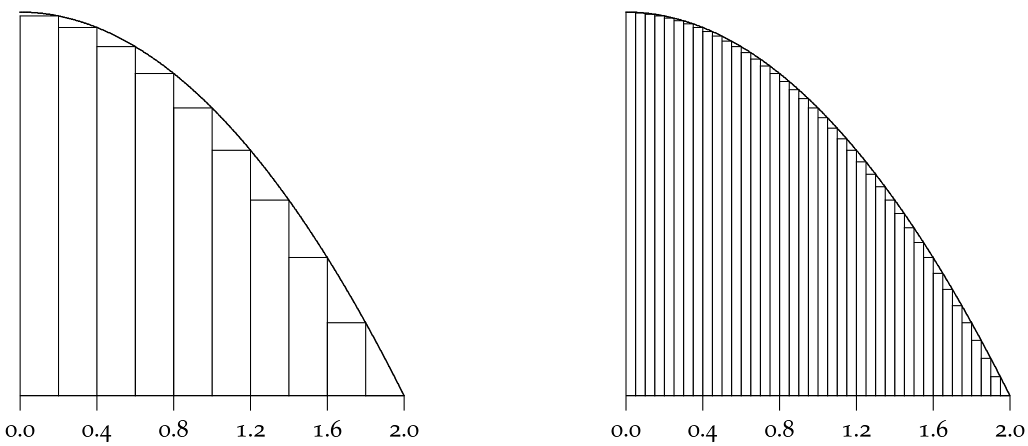

Figure 2.2: The left-hand graph shows for the function $f(x) = 2 - \frac{1}{2}x^2$ on $[0, 2]$ with the interval divided in to $n = 10$ subintervals of width $\frac{2}{n} = \frac{2}{10}$. The heights of the rectangles come from the function values $f\left(\frac{2i}{n}\right) = f\left(\frac{2i}{10}\right)$ as i changes from 1 to 10. he sum of the areas of the 10 rectangles is 2.46 and is an approximation of the area under the curve on the interval $[0, 2]$.

In the right-hand graph, $n = 40$, and the same process is repeated. The sum of the areas of all the rectangles is a better approximation of the true area under the curve on the interval $[0, 2]$.

By letting $n \to \infty$, we may obtain the exact area under the curve. We found that this area was $\frac{8}{3}$ in (2.2).