Taylor Polynomials

Let $f(x) = e^x$ and let $p(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$. Without using a calculator, evaluate $f(1.2)$ and $p(1.2)$. Ok, I'm still waiting.... With a little effort it is possible to evaluate

$$p(1.2) = 1 + 1.2 + \tfrac{1}{2}(1.44) + \tfrac{1}{6}(1.728) = 3.208.$$

However $f(1.2) = e^{1.2}$, hmm, not so much. When we use a calculator we find

$$e^{1.2} = 3.3201169227.$$

How did your calculator determine this value? Remember e is an irrational number whose value is not known. How does your calculator compute e^{π}?

This situation is the whole point of trying to find ways to approximate functions that are not easily calculated by hand. These include a lot of the familiar calculus functions: e^x, $\ln x$, the trig functions and their inverses such as $\sin x$, $\cos x$, $\arcsin x$, $\arctan x$, and even more familiar functions such as $\sqrt{x}$ and more generally $\sqrt[n]{x}$. It would be nice to be able to approximate such functions by polynomials, since polynomials are simpler to calculate.

The good news is that this can be done (rather simply) using what are called Taylor polynomials. Now we don't have a lot of time to spend doing this, so I will just present a few facts and a simple example. Your text has more examples and details which you should look at.

The Tangent Line

Recall that if we have a function $y = f(x)$ that is differentiable at $x = a$, then the slope of the tangent line at a is $m = f'(a)$. To obtain the equation of the tangent line to f at the point $(a, f(a))$ we use this slope

$$m = f'(a) = \frac{y - f(a)}{x - a}$$

and solve for y:

$$y =$$

Geometrically the tangent line is the unique line that passes through the point $(a, f(a))$ and *has the same slope* as the curve $y = f(x)$ at that point. It is the best linear approximation to the curve. Let's call the equation of the tangent line $p_1(x)$, to indicate that it is a polynomial and our first approximation. Figure 15.1 shows a typical example of this situation.

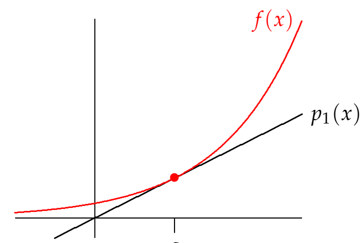


Figure 15.1: The tangent line $p_1(x)$ is the unique line that passes through the point $(a, f(a))$ and *has the same slope* as the curve $y = f(x)$ at that point. It is the best linear approximation to the curve.

To make the approximation even better we could insist that *both* the first and the second derivative of our approximation match those of the original function. Let's call this new approximation $p_2(x)$. Geometrically this means that we want the *slope* *and concavity* of p_2 to be the same as f at $x = a$. To create such a function we need to add a quadratic (square) term to our first approximation.

$$p_2(x) = \underbrace{f(a) + f'(a)\cdot(x-a)}_{p_1(x)} + \underbrace{c(x-a)^2}_{\text{quadratic term}}$$

where c is some constant. But which?

We can figure out what c has to be using the following process. Calculate the first and second derivatives of p_2

$$p_2'(x) =$$
$$p_2''(x) =$$

If we evaluate these derivatives at $x = a$, we get

$$p_2'(a) =$$
$$p_2''(a) =$$

So the first derivative still matches the first derivative of f at a. We need the second derivative to be $f''(a)$. In other words we need

$$f''(a) = p''(a) = \qquad\qquad \Rightarrow c =$$

So this means

$$p_2(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

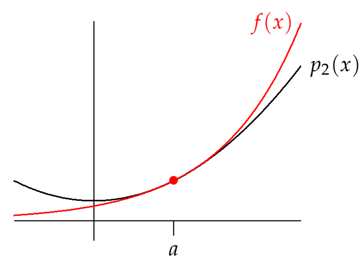


Figure 15.2: The function $p_2(x)$ is the unique degree two polynomial through the point $(a, f(a))$ that has the same first and second derivatives (slope and concavity) as f at that point.

Figure 15.2 shows illustrates this situation.

Let's do one further approximation p_3 that requires the third derivative (the jerk) to match that of f at $x = a$. This time we have

$$p_3(x) = \underbrace{f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2}_{p_2(x)} + \underbrace{d(x-a)^3}_{\text{cubic term}}.$$

Notice that the first three derivatives of p_3 are

$$p_3'(x) = f'(a) + f''(a)(x-a) + 3d(x-a)^2$$
$$p_3''(x) = f''(a) + 3\cdot 2\cdot d(x-a)^1$$
$$p_3'''(x) =$$

If we evaluate these derivatives at $x = a$, we get

$$p_3''(a) = f'(a)$$
$$p_3''(a) = f''(a)$$
$$p_3'''(a) = \qquad\qquad \overset{\text{which we want}}{=} f'''(a)$$

The first two derivatives still match those of f at a. We need the third derivative to be $f'''(a)$. In other words, we need $f'''(a) = 3!d$ or $d = f'''(a)/3!$. So this means

$$p_3(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3.$$

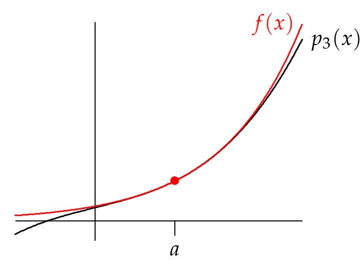


Figure 15.3: The function $p_3(x)$ is the unique degree three polynomial through the point $(a, f(a))$ that has the same first three derivatives as f there.

Figure 15.3 shows that $p_2(x)$ is an even better approximation of the original curve.

Do you see a pattern? If we use that fact that $2! = 2$, then we can write the last approximation as

$$p_3(x) = f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3.$$

We could keep going with these approximations. After doing this process n times we would get a degree n polynomial whose first n derivatives agree with those of f at $x = a$.

DEFINITION 15.1 (Taylor Polynonials)**.** Assume f has at least n derivatives. The **nth order Taylor polynomial** for f at $x = a$ is

$$p_n(x) = f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The point a is called the **center** of the expansion. $p_n(x)$ has the same first n derivatives as $f(x)$ at $x = a$. The nth order Taylor polynomial can also be written as

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

[Remember that $0! = 1$ and that $f^{(0)}(a) = f(a)$.]

Writing a Taylor polynomial using summation notation can't help but bring series to mind. In fact the Taylor *series* for f centered at a is the 'infinite degree' polynomial

$$p_\infty(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Now there are all sorts of questions we can and will ask about such expressions (convergence being the most important). But it turns out under the right circumstances that such series are not just approximations to f, they are actually *equal* to f in some interval around a. In other words, $f(x) = p_\infty(x)$ and this gives us a way to calculate functions such as e^x or $\ln x$ that we can't ordinarily evaluate.

Working with Taylor Polynomials

We won't spend a lot of time calculating Taylor polynomials (or series). But here's a fun example that illustrates the idea.

EXAMPLE 15.0.1. Let's take the function $f(x) = e^{x/2}$ and let's use $a = 0$ as the center. Determine the Taylor polynomials $p_1(x)$, $p_2(x)$, and $p_3(x)$.

Use this pattern to determine the general order n Taylor polynomial $p_n(x)$ for $e^{x/2}$.

Then use $p_3(x)$ to approximate $e^{0.2}$. (Be careful, use $x = 0.4$.)

SOLUTION. By Definition 15.1 the formula for $p_n(x)$ with center $a = 0$ is given by

$$p_n(x) = f(0) + f'(0) \cdot (x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(n)}(0)}{n!}(x-0)^n.$$

So we need to calculate the derivatives of $f(x) = e^{x/2}$ and then evaluate them at 0. Well,

$$\begin{aligned} f(x) &= & f(0) &= \\ f'(x) &= & f'(0) &= \\ f''(x) &= & f''(0) &= \\ f'''(x) &= & f''(0) &= \\ &\vdots \\ f^{(k)}(x) &= & f'^{(k)}(0) &= \end{aligned}$$

In particular, we see that the first three Taylor polynomials for $e^{x/2}$ centered at 0 are

$$\begin{aligned} p_1(x) &= 1 + \frac{x}{2^1 \cdot 1!} = 1 + \frac{x}{2} \\ p_2(x) &= \\ p_3(x) &= \end{aligned}$$

and

$$\begin{aligned} p_n(x) &= f(0) + f'(0) \cdot (x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(n)}(0)}{n!}(x-0)^n \\ &= \\ &= \sum_{k=0}^{n} \frac{\qquad\qquad}{} x^k \end{aligned}$$

The graphs of these functions are shown in the figure on the next page. Notice how the approximation to $e^{x/2}$ improves as the order of the polynomial increases.

Let's do the approximation of $e^{0.2}$. Since our function is $e^{x/2}$ we will need to use $x = 0.4$:

$$\begin{aligned} e^{0.2} = e^{0.4/2} \approx p_3(0.4) &= 1 + \frac{0.4^1}{2^1 \cdot 1!} + \frac{0.4^2}{2^2 \cdot 2!} + \frac{0.4^3}{2^3 \cdot 3!} \\ &= 1 + \frac{0.4}{2} + \frac{0.16}{8} + \frac{0.064}{48} \\ &= 1.2213333333 \end{aligned}$$

Compare to the calculator value: $e^{0.2} = 1.21402$. So $p_3(.4)$ is a pretty good estimate.

In this case it is easy to see that the 'Taylor series' for $e^{x/2}$ is the 'infinite degree' polynomial

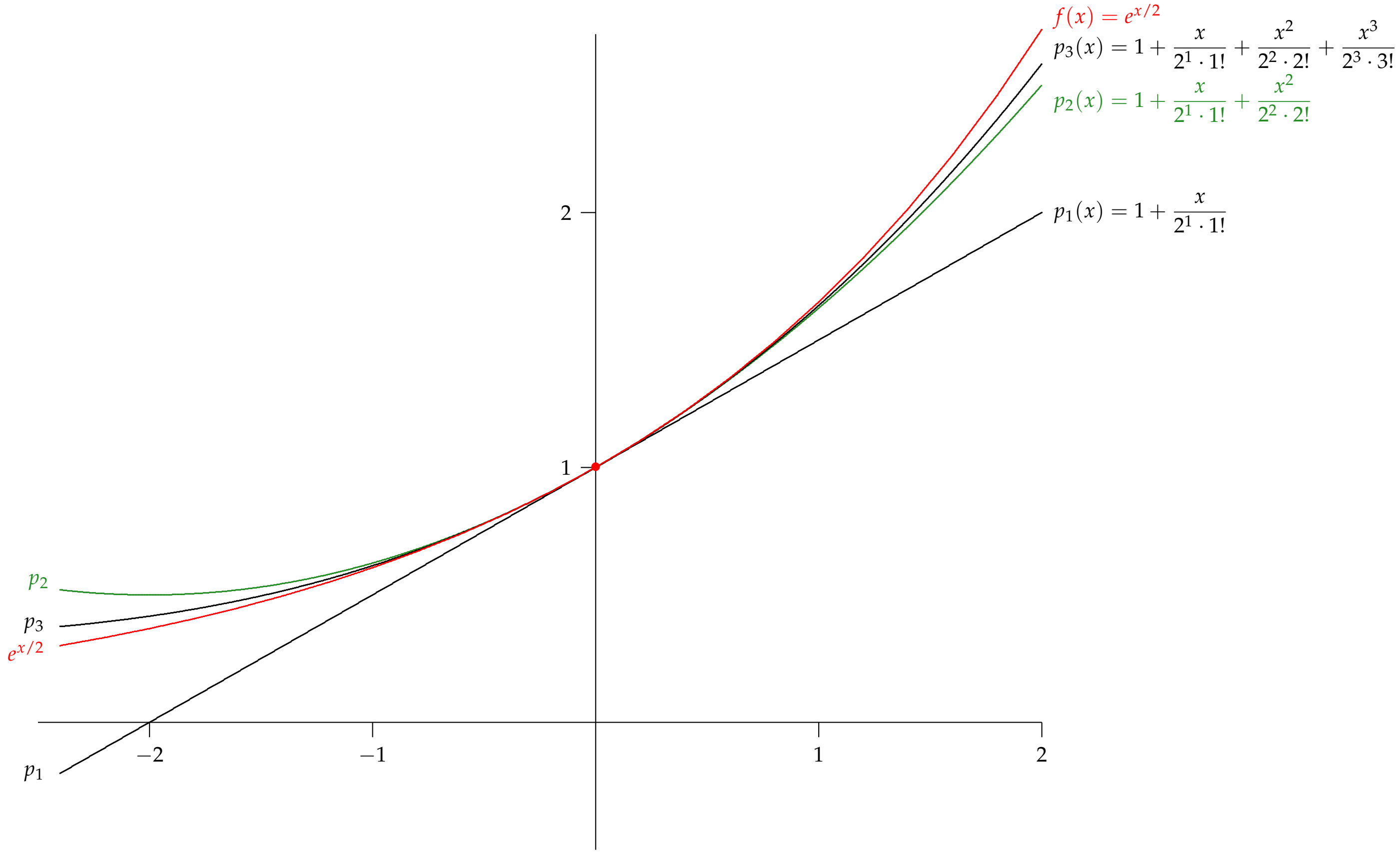
$$p_\infty(x) = 1 + \frac{x}{2^1 \cdot 1!} + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \cdots + \frac{x^n}{2^n \cdot n!} + \cdots$$

or expressed as a series

$$p_\infty(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n \cdot n!}.$$

Each number x can be 'plugged into' the series and we can determine whether $p_\infty(x)$ exists (whether the series converges for this value of x) or does not exist (the series diverges). If we do this for each and every x, we can determine the domain of $p_\infty(x)$. We will soon see that this can be done very efficiently. Moreover, it is possible to show that in this case $e^{x/2}$ actually equals $p_\infty(x)$. This gives us a mechanical way to calculate $e^{x/2}$.

The ultimate goal is to represent familiar functions like $\sin x$ and e^x (which are done as examples in your text) as 'infinite degree' polynomials so that the values of these functions can be estimated quickly. This is how your calculator works!



The first three Taylor polynomials for $f(x) = e^{x/2}$ centered at 0. Notice how the approximation improves as n increases.

EXAMPLE 15.0.2. Let's take the function $f(x) = \sin x$ and let's use $a = 0$ as the center. Determine the Taylor polynomials $p_1(x), \ldots, p_7(x)$.

EXAMPLE 15.0.3. Let's take the function $f(x) = \ln x$ and let's use $a = 1$ as the center. Determine the Taylor polynomials $p_1(x), p_2(x), p_3(x)$.