8.2 *Doing It Twice: One Good Turn Deserves Another.*

There are a couple of situations where using integration by parts twice is just the ticket. Part way through you should be able to see why this works.

EXAMPLE 8.2.1 (Parts: Reduction). Determine $\displaystyle\int x^2 \sin x \, dx$.

SOLUTION. The two functions in the integrand, x^2 and $\sin x$ are unrelated to each other. Parts should come to mind. Since $\sin x$ is the more complicated piece and can be integrated, use it as dv. Likewise, x^2 becomes simpler when integrated ($\sin x$ does not), so use it as u. So (being careful with signs)

$u = x^2$	$dv = \sin x \, dx$	$\int u \, dv = uv - \int v \, du$
$du = 2x \, dx$	$v = \int dv = \int \sin x \, dx = -\cos x$	$\int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx$

Notice that the new integral $\int 2x \cos x \, dx$ is not immediately 'doable' but is simpler than the original one (the power of x is lower) and can be done with parts.

$u = 2x$	$dv = \cos x \, dx$	$\int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx$
$du = 2 \, dx$	$v = \int dv = \int \cos x \, dx = \sin x$	$= -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx$
		$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$

Check that this is correct by differentiating the answer.

$$\begin{aligned}\frac{d}{dx}\left(-x^2 \cos x + 2x \sin x + 2 \cos x + c\right) &= -2x \cos x + x^2 \sin x + 2 \sin x + 2x \cos x - 2 \sin x \\ &= x^2 \sin x,\end{aligned}$$

which is the original integrand.

EXAMPLE 8.2.2 (Parts: Circular Reasoning). Determine $\displaystyle\int e^x \sin x \, dx$.

SOLUTION. The two functions in the integrand, e^x and $\sin x$ are unrelated to each other. Think parts! Here it really does not matter which we use as u or dv, though I would give a slight preference to using $u = \sin x$ since it is slightly easier to differentiate $\sin x$ than it is to integrate it (where we get a negative sign). So just to be contrary I will do it the other way. We will use parts twice, as it turns out.

$u = e^x$	$dv = \sin x \, dx$	$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$
$du = e^x dx$	$v = -\cos x$	Watch the signs! This is what I meant above.

Notice that the new integral $\int e^x \cos x \, dx$ is not a whole lot different than the first one. Parts is indicated. **Important:** Continue to set up the parts in the same way: u is still the exponential function and dv is still the trig function.

$u = e^x$	$dv = \cos x \, dx$	$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$. Signs!
$du = e^x dx$	$v = \sin x$	$\int e^x \sin x \, dx$ appears on both sides of the equation.
		Solve for it: $2 \int e^x \sin x \, dx = e^x(\sin x - \cos x) + c$
		Thus, $\int e^x \sin x \, dx = \frac{1}{2} e^x(\sin x - \cos x) + c$

You should check that this is correct by differentiating the answer. Notice what happened here. We *never* actually did one of the integrals. We kept applying the parts formula until the problem more or less circled around on itself. In these sorts of problems you must be very careful with the signs attached to the various integrals. These are great problems to demonstrate your mastering of this technique.

EXAMPLE 8.2.3 (Udoable Becomes Doable). Determine $\int \arcsin x\, dx$. Sometimes we simply don't know an antiderivative for a familiar function. Parts can be a way to solve the problem (see Example 8.1.3 for another example of this).

SOLUTION. We don't have much choice. If parts applies we must let $u = \arcsin x$ since we do not know how to integrate it. So

$u = \arcsin x$	$dv = dx$	$\int \arcsin x\, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx$
$du = \frac{1}{\sqrt{1-x^2}}\, dx$	$v = \int dv = \int dx = x$	Substitution $u = 1 - x^2$ so $du = -2x\, dx \Rightarrow \frac{1}{2}\, du = -x\, dx$
		$\int \arcsin x\, dx = x \arcsin x + \int \frac{1}{2} u^{-1/2}\, du$ Signs!!
		$\int \arcsin x\, dx = x \arcsin x + \sqrt{u} + c = x \arcsin x + \sqrt{1-x^2} + c$

You should check that this is correct by differentiating the answer. Another great problems to demonstrate your understanding of integration techniques.

EXAMPLE 8.2.4 (Tricky). Determine $\displaystyle\int \frac{xe^{2x}}{(2x+1)^2}\, dx$.

SOLUTION. It does not look like a substitution problem, so it is probably a parts problem. Here the choice of parts is tricky. But if we keep in mind the suggestion that we use the "most complicated portion of the integrand that you can integrate for dv", then let's use $dv = \frac{1}{(2x+1)^2}\, dx$. Then by the power rule and a 'mental adjustment',

$$v = \int (2x+1)^{-2}\, dx = -\frac{1}{2}(2x+1)^{-1} = -\frac{1}{2(2x+1)}.$$

Now u is the rest of the integrand so $u = xe^{2x}$ and

$$du = (2xe^{2x} + e^{2x}\, dx = (2x+1)e^{2x}\, dx.$$

Note the factor of $2x+1$. Putting this all together (and keeping track of signs),

$$\begin{aligned}
\int \frac{xe^{2x}}{(2x+1)^2}\, dx = uv - \int v\, du &= -\frac{xe^{2x}}{2(2x+1)} + \int \frac{(2x+1)e^{2x}}{2(2x+1)}\, dx \\
&= -\frac{xe^{2x}}{2(2x+1)} + \int \frac{e^{2x}}{2}\, dx \\
&= -\frac{xe^{2x}}{2(2x+1)} + \frac{e^{2x}}{4} + c
\end{aligned}$$

where we used a 'mental adjustment' at the last step. You should check that this is correct by differentiating the answer.

8.3 Problems

1. Integral Mix Up: Before working these out, go through and classify each by the technique that you think will apply: substitution, parts, parts twice, or ordinary methods. Which can't you do yet? The **answers** are below.

(a) $\int 2e^{-\pi x}\, dx$	(b) $\int \cos x e^{\sin x}\, dx$	(c) $\int e^x \cos x\, dx$
(d) $\int x \cos x\, dx$	(e) $\int \cos(2\pi x)\, dx$	(f) $\int \frac{\ln x}{x}\, dx$
(g) $\int (x^2+1)e^{x^3+3x}\, dx$	(h) $\int (x^2+1)e^x\, dx$	(i) $\int x^2 \ln x\, dx$
(j) $\int \sec^2(2x)\, dx$	(k) $\int \frac{x}{25+x^2}\, dx$	(l) $\int \frac{1}{1+25x^2}\, dx$
(m) $\int \frac{1}{\sqrt{1-9x^2}}\, dx$	(n) $\int \frac{\cos x}{\sqrt{1-\sin^2 x}}\, dx$	(o) $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$

1. Some Answers to the Mix-Up Problem: (All "$+c$".)

(*a*) $\dfrac{-2e^{-\pi x}}{\pi}$	(*b*) $e^{\sin x}$	(*c*) $\frac{1}{2}e^x(\cos x + \sin x)$
(*d*) $x \sin x + \cos x$	(*e*) $\frac{1}{2\pi}\sin(2\pi x)$	(*f*) $\dfrac{(\ln x)^2}{2}$
(*g*) $\frac{1}{3}e^{x^3+3x}$	(*h*) $(x^2 - 2x + 3)e^x$	(*i*) $\frac{1}{3}x^3\left(\ln x - \frac{1}{3}\right)$
(*j*) $\frac{1}{2}\tan(2x)$	(*k*) $\frac{1}{2}\ln(25 + x^2)$	(*l*) $\frac{1}{5}\arctan 5x$
(*m*) $\frac{1}{3}\arcsin 3x$	(*n*) $\arcsin(\sin x)$	(*o*) $\frac{1}{2}(\arcsin x)^2$

Parts: Further Examples

We end with a few more examples that involve trig functions and that provide a segue to our foray into trig integrals more generally.

EXAMPLE 8.3.1 (Oddball). Determine $\int \cos(\ln x)\, dx$. Careful! This is not a product of functions, it is a composition.

SOLUTION. Again we don't have much choice. It's not substitution since the argument[1] is $u = \ln x$ and du is nowhere in sight. So let's try parts. Since the integrand consists of a single function that we don't know how to antidifferentiate we must let $u = \cos(\ln x)$. So

[1] The term 'argument' means the input to the function

$u = \cos(\ln x)$	$dv = dx$	$\int \cos(\ln x)\, dx = x\cos(\ln x) + \int x \cdot \frac{1}{x}\sin(\ln x)\, dx$ Sign!
$du = -\frac{1}{x}\sin(\ln x)\, dx$	$v = x$	$= x\cos(\ln x) + \int \sin(\ln x)\, dx$

The new integral $\int \sin(\ln x)\, dx$ looks much like the old one, so we try parts again and hope that we can 'circle around' to where we started as in Example 8.2.2.

$u = \sin(\ln x)$	$dv = dx$	$\int \sin(\ln x)\, dx = x\sin(\ln x) - \int x \cdot \frac{1}{x}\cos(\ln x)\, dx$
$du = \frac{1}{x}\cos(\ln x)\, dx$	$v = x$	$= x\sin(\ln x) - \int \cos(\ln x)\, dx$ Cycles back

Putting the two parts integrals together:

$$\int \cos(\ln x)\, dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\, dx$$

which gives

$$2\int \cos(\ln x)\, dx = x\cos(\ln x) + x\sin(\ln x)$$

or

$$\int \cos(\ln x)\, dx = \frac{1}{2}\left(x\cos(\ln x) + x\sin(\ln x)\right) + c$$

You should check that this is correct by differentiating the answer.

EXAMPLE 8.3.2 (Trig). Determine $\int \sin(x)\cos(4x)\, dx$. Careful! How does this problem differ from $\int \sin(4x)\cos(4x)\, dx$? The latter we can do by substitution.

SOLUTION. It's not substitution, so we try parts. It is a little easier to integrate $\sin x$ rather than $\cos(4x)$ because of the constants involved. So

$u = \cos(4x)$	$dv = \sin x, dx$	$\int \sin(x)\cos(4x)\, dx = -\cos(x)\cos(4x) - \int \cos(x)4\sin(4x)\, dx$
$du = -4\sin(4x)\, dx$	$v = -\cos x$	Parts again! Should circle around.

The new integral $-\int \cos(x)4\sin(4x)\, dx$ looks much like the old one, so we try parts again and hope that we can 'circle around' to where we started as in Example 8.2.2.

Remember to choose the parts in the same way at each stage.

$$\boxed{\begin{array}{ll} u = 4\sin(4x) & dv = \cos x\,dx \\ du = 16\cos(4x)\,dx & v = \sin x \end{array}}$$

So

$$\begin{aligned}\int \sin(x)\cos(4x)\,dx &= -\cos(x)\cos(4x) - \int \cos(x)4\sin(4x)\,dx \\ &= -\cos(x)\cos(4x) - \left[4\sin(x)\sin(4x) - 16\int \sin(x)\cos(4x)\,dx\right].\end{aligned}$$

This gives

$$-15\int \sin(x)\cos(4x)\,dx = -\cos(x)\cos(4x) - 4\sin(x)\sin(4x)$$

or

$$\int \sin(x)\cos(4x)\,dx = \frac{\cos(x)\cos(4x) - 4\sin(x)\sin(4x)}{15} + c$$

Not too bad, but be careful of your signs throughout similar problems. You should check that this is correct by differentiating the answer.

Integrating Low Powers of the Secant Function. Remember that we were able to solve $\int \sec x\,dx$ using substitution. We saw

$$\boxed{\int \sec x\,dx = \ln|\sec x + \tan x| + c.}$$

Of course

$$\boxed{\int \sec^2 x\,dx = \tan x + c.}$$

EXAMPLE 8.3.3 (Trig). What about

$$\int \sec^3 x\,dx?$$

SOLUTION. Following the suggestions for integration by parts, the most complicated factor in the integrand that we can integrate is $dv = \sec^2 x\,dx$. Working this out we get

$$\begin{array}{ll} u = \sec x & dv = \sec^2 x\,dx \\ du = \sec x\tan x\,dx & v = \tan x \end{array}$$

So

$$\begin{aligned}\int \sec^3 x\,dx = \int \sec x\sec^2 x\,dx &= \sec x\tan x - \int \sec x\tan^2 x\,dx \\ &= \sec x\tan x - \int \sec x(\sec^2 x - 1)\,dx \\ &= \sec x\tan x - \int \sec^3 x - \sec x\,dx\end{aligned}$$

Solving for the original integral we get

$$2\int \sec^3 x\,dx = \sec x\tan x + \int \sec x\,dx = \sec x\tan x + \ln|\sec x + \tan x|.$$

So

$$\boxed{\int \sec^3 x\,dx = \frac{\sec x\tan x + \ln|\sec x + \tan x|}{2} + c.}$$

We will look at some additional trig integrals in the next section.