## 6.4 *Volumes of Revolution: The Shell Method*

In this section we will derive an alternative method—called the shell method—for calculating volumes of revolution. This method will be easier than the disk method for some problems and harder for others. There are also some problems that we cannot do with the disk method that become possible with the shell method. We will again use the 'subdivide and conquer' strategy with Riemann sums to derive the appropriate integral formula. Our objectives are

- to develop the volume formula for solids of revolution using the shell method;
- to compare and contrast the shell and disk methods.

We start with a continuous function $y = f(x)$ on $[a, b]$. We create a regular partition of $[a, b]$ using $n$ intervals and draw the corresponding approximating rectangles of equal width $\Delta x$. In left half of Figure 6.26 we have drawn a single representative approximating rectangle on the $i$th subinterval. This time we rotate the rectangle about the $y$-axis, not the $x$-axis. (Compare the right half of Figure 6.26 to the right half of Figure 6.11.)

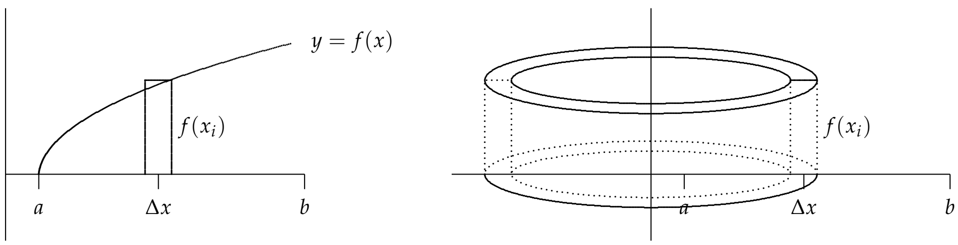


Figure 6.26: Left: The region under the continuous curve $y = f(x)$ on the interval $[a, b]$ and a representative rectangle. Right: The solid shell of outer radius $r = x_i$, height $h = f(x_i)$, and 'wall width' $\Delta x$ generated by rotating a representative rectangle about the $y$-axis.

When the rectangle in Figure 6.26 is rotated around the $y$-axis, a hollow thin-walled cylinder called a 'shell' is created. We want to determine the volume of the solid part of the cylinder (not the volume of the hollow part). A cute way to do this is to take to take the shell and slice it vertically and then 'unroll it' so that it forms a flat slab as in Figure 6.27.

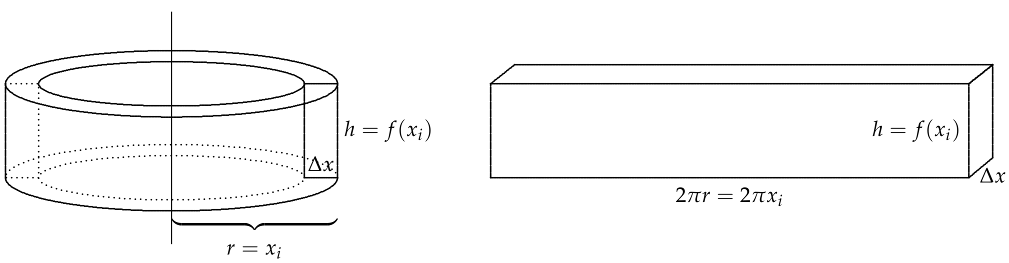


Figure 6.27: Left: The solid shell of outer radius $r = x_i$, height $h = f(x_i)$, and 'wall width' $\Delta x$ generated by rotating the representative rectangle about the $y$-axis. Right: The shell cut open and laid out as a slab of length $2\pi r = 2\pi x_i$, height $h = f(x_i)$, and width $\Delta x$. The volume of the shell and slab are equal: $V_i = 2\pi x f(x_i)\Delta x$.

The resulting 'slab' has the same height as the shell, $h = f(x_i)$ and the same width $w = \Delta x$. The the length of the slab is the same as the circumference of the cylinder. Since the cylinder has (outer) radius $r = x_i$, the circumference of the cylinder is $2\pi r = 2\pi x_i$. Since the slab is really a thin rectangular box, its volume is $V = \text{length} \times \text{width} \times \text{height}$. Using Figure 6.27, this translates into

$$\text{Volume of a representative shell} = V_i = lwh = 2\pi x_i f(x_i)\Delta x.$$

Approximating the volume of the entire solid by $n$ such shells of width $\Delta x$ and height $f(x_i)$ produces a Riemann sum

$$\text{Volume of Revolution} \approx \sum_{i=1}^{n} 2\pi x_i f(x_i)\Delta x = 2\pi \sum_{i=1}^{n} x_i f(x_i)\Delta x. \tag{6.6}$$

As usual, to improve the approximation we let the number of subdivisions $n \to \infty$ and take a limit. Recall from our earlier work with Riemann sums, this limit exists because $xf(x)$ is continuous on $[a, b]$ since $x$ and $f(x)$ are both continuous there. So

$$\text{Volume of Revolution by Shells} = \lim_{n\to\infty} 2\pi \sum_{i=1}^{n} x_i f(x_i)\Delta x = 2\pi \int_a^b x f(x)\, dx \tag{6.7}$$

where we have used the fact that the limit of a Riemann sum is a definite integral. Of course, we could use this same process if we rotated the region about the $x$-axis and integrated along the $y$-axis.

*Stop!* Notice how we used the 'subdivide and conquer' process to approximate the quantity we wish to determine. That is we have subdivided the volume into 'approximating shells' whose volume we know how to compute. We have then refined this approximation by using finer and finer subdivisions. Taking the limit of this process provides the answer to our question. Identifying that limit with an integral makes it possible to compute the volume in question. We have proved

**THEOREM 6.3 (The Shell Method).** If $V$ is the volume of the solid of revolution determined by rotating the continuous function $f(x)$ on the interval $[a, b]$ about the $y$-axis, then

$$V = 2\pi \int_a^b x f(x)\, dx. \tag{6.8}$$

If $V$ is the volume of the solid of revolution determined by rotating the continuous function $f(y)$ on the interval $[c, d]$ about the $y$-axis, then

$$V = 2\pi \int_c^d y f(y)\, dy. \tag{6.9}$$

*Note:* The axis of rotation and the variable of integration are not the same in the shell method, e.g., when rotating around the $y$-axis, the integration takes place along the $x$-axis. This differs from the disk method where the axis of rotation and axis of integration are the same.

## *Examples*

We'll do several examples to see how the shell method works and compares with the disk method.

**EXAMPLE 6.13.** Consider the region enclosed by the curves $y = f(x) = x^3 + x$, $x = 2$, and the $x$-axis. Rotate the region about the $y$-axis and find the resulting volume.

**SOLUTION.** We use the shell method because the rotation is about the $y$-axis. If we used the disk method, we would need to solve for $x$ in terms of $y$. This is not easily done here (and, in fact, would likely be impossible for you). This is one of the most important advantages of the shell method: Inverse functions are not required if the function variable and axis of integration (not the axis of rotation) are the same. A sketch of the region and a representative rectangle appears in Figure 6.28.

Using Theorem 6.3

$$\begin{aligned} V = 2\pi \int_a^b x f(x)\, dx = 2\pi \int_0^2 x(x^3 + x)\, dx &= 2\pi \int_0^2 x^4 + x^2\, dx \\ &= 2\pi \left( \frac{x^5}{5} + \frac{x^3}{3} \right) \Big|_0^2 \\ &= 2\pi \left( \frac{32}{5} + \frac{8}{3} - 0 \right) \\ &= \frac{292\pi}{15}. \end{aligned}$$

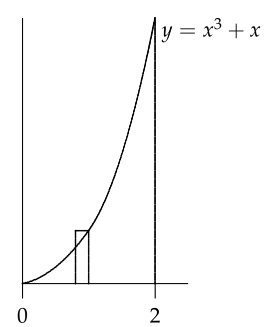


Figure 6.28: The region enclosed by the curves $y = f(x) = x^3 + x$, $x = 2$, and the $x$-axis. Rotate the region about the $y$-axis and find the resulting volume.

**EXAMPLE 6.14.** Let $R$ be region enclosed by the curves $y = f(x) = 2x$ and $y = \frac{x^2}{2}$. Rotate $R$ about the $y$-axis and find the resulting volume.

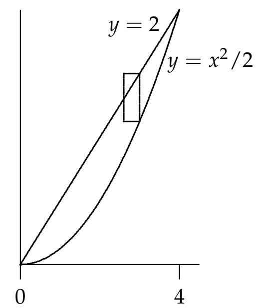


Figure 6.29: The region enclosed by the curves $y = f(x) = 2x$, $y = \frac{x^2}{2}$. Rotate the region about the $y$-axis and find the resulting volume.

**SOLUTION.** The two curves meet when

$$\frac{x^2}{2} = 2x \Rightarrow \frac{x^2}{2} - x = x\left(\frac{x}{2} - 1\right) = 0 \Rightarrow x = 0,\ 2.$$

A sketch of the region and a representative rectangle appears in Figure 6.29.

If we used the disk method, we would need to solve for $x$ in terms of $y$ and we would need to use two integrals. Using the shell method, however, we can treat the height of the cylinder as the difference in height between the two curves: $2x - \frac{x^2}{2}$. So by (Theorem 6.3)

$$\begin{aligned} V = 2\pi \int_a^b x f(x)\,dx = 2\pi \int_0^4 x\left(2x - \frac{x^2}{2}\right)\,dx &= 2\pi \int_0^4 2x^2 - \frac{x^3}{2}\,dx \\ &= 2\pi \left(\frac{2x^3}{3} - \frac{x^4}{8}\right)\Bigg|_0^4 \\ &= 2\pi\left(\frac{128}{3} - 32 - 0\right) \\ &= \frac{64\pi}{3}. \end{aligned}$$

**YOU TRY IT 6.26.** Try Example 6.14(b) using the disk method. Which method did you find easier?

**EXAMPLE 6.15.** Let $R$ be region enclosed by the curves $y = f(x) = \arctan x$ and $y = \frac{\pi}{4}$ and the $y$-axis. Rotate $R$ about the $y$-axis and find the resulting volume.

**SOLUTION.** The two curves meet when

$$\arctan x = \frac{\pi}{4} \Rightarrow x = 1.$$

A sketch of the region and a representative rectangle appears in Figure 6.30. Using shells, the volume is

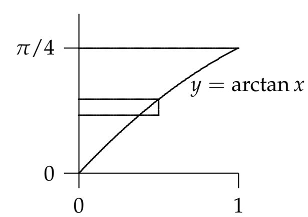


Figure 6.30: The region enclosed by the curves $y = f(x) = \arctan x$, $y = \frac{\pi}{4}$ and the $y$-axis. Rotate the region about the $y$-axis and find the resulting volume. Which method is most appropriate: shells or disks? The representative rectangle is for the disk method.

$$V = 2\pi \int_a^b x f(x)\,dx = 2\pi \int_0^4 x\,(1 - \arctan x)\,dx = 2\pi \int_0^4 x - x\arctan x\,dx = ???$$

We don't know an antiderivative for $x \arctan x$.

If we use the disk method, we need to solve for $x$ in terms of $y$. Here $y = \arctan x$ so $x = \tan y$. A representative rectangle is horizontal and is shown in Figure 6.30. By Theorem 6.2

$$\begin{aligned} V = \pi \int_a^b [g(y)]^2\,dy = \pi \int_0^{\pi/4} (\tan y)^2\,dy &= \pi \int_0^{\pi/4} \sec^2 y - 1\,dy \\ &= \pi\,(\tan y - y)\Bigg|_0^{\pi/4} \\ &= \pi\left(1 - \frac{\pi}{4} - 0\right) \\ &= \pi - \frac{\pi^2}{4}. \end{aligned}$$

We used the trig identity $\tan^2 y = \sec^2 y - 1$ in the work above. This example shows that it is important to be familiar with both the shell and disk methods. One or the other may be most appropriate in a given situation.

**EXAMPLE 6.16.** Let $R$ be region enclosed by the curves $y = f(x) = e^{x^2}$, $x = 1$, and the $x$-axis. Rotate $R$ about the $y$-axis and find the resulting volume.

**SOLUTION.** A sketch of the region and a representative rectangle appears in Figure 6.31. Using shells, the volume is

$$\begin{aligned} V &= 2\pi \int_a^b x f(x)\,dx \\ &= 2\pi \int_0^1 x e^{x^2}\,dx \qquad\qquad (\text{let } u = x^2) \\ &= \pi \int_0^1 e^u\,du = \pi\,(e-1)\,. \end{aligned}$$

Notice we needed to use a simple $u$-substitution: $u = x^2$, $du = 2x\,dx \Rightarrow \pi\,du = 2\pi x\,dx$, and $x = 0 \Rightarrow u = 0$; $x = 1 \Rightarrow u = 1$.

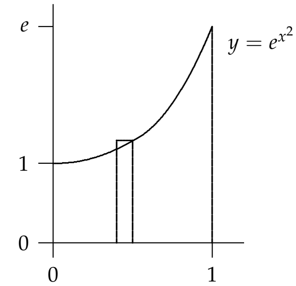


Figure 6.31: The region enclosed by the curves $y = f(x) = e^{x^2}$, $x = 1$, and the $x$-axis. Rotate the region about the $y$-axis and find the resulting volume.

**YOU TRY IT 6.27.** Try Example 6.16 using the disk method. Is it possible?

**YOU TRY IT 6.28.** Try setting up the integrals for Example 6.17 using the disk method. Is it easy?

We end with an example using the shell method for a rotation about the $x$-axis.

**EXAMPLE 6.17.** Let $R$ be region enclosed by the curves $x = 1-(y-1)^2$ and the $y$-axis. Rotate $R$ about the $x$-axis and find the resulting volume.

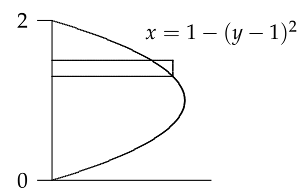


Figure 6.32: The region enclosed by the curves $x = 1-(y-1)^2$ and the $y$-axis. Rotate the region about the $x$-axis and find the resulting volume.

**SOLUTION.** The sideways parabola along with a representative rectangle is shown in Figure 6.32. Using shells and integration along the $y$-axis, the volume is

$$\begin{aligned} V = 2\pi \int_c^d y f(y)\,dy = 2\pi \int_0^2 y(1-(y-1)^2)\,dy &= 2\pi \int_0^2 y(2y-y^2)\,dy \\ &= 2\pi \int_0^2 2y^3 - y^2\,dy \\ &= 2\pi\left(\frac{y^4}{2} - \frac{y^3}{3}\right) = 2\pi\left(8 - \frac{8}{3} - 0\right) = \frac{32\pi}{3}. \end{aligned}$$

**YOU TRY IT 6.29 (Using the Shell Method).** **Just set up the integrals** for each of the following volume problems. Simplify the integrands where possible.

(*a*) $R$ is the region enclosed by $y = x^2$, $y = x+2$, and the $y$ axis in the first quadrant. Rotate $R$ about the $y$-axis.

(*b*) $S$ is the region enclosed by $y = -x^2+2x+3$, $y = 3x-3$, the $y$ axis, and the $x$ axis in the first quadrant. Rotate $S$ about the $y$-axis.

(*c*) $T$ is the region enclosed by $y = -x^2+2x+3$, $y = 3x-3$, and the $x$ axis in the first quadrant. Rotate $T$ about the $y$-axis.

(*d*) $S$ is the region enclosed by $y = \frac{1}{2}x^2+2$ and $y = x^2$ in the first quadrant. Rotate $S$ about the $y$-axis.

(*e*) $T$ is the region enclosed by $y = \sqrt{x-2}$, $y = 4-x$, the $y$-axis and the $x$-axis. Rotate $T$ about the $y$-axis.

(*f*) $V$ is the region enclosed by $y = \sqrt{x-2}$, $y = 4-x$, and the $x$-axis. Rotate $V$ about the $y$-axis.

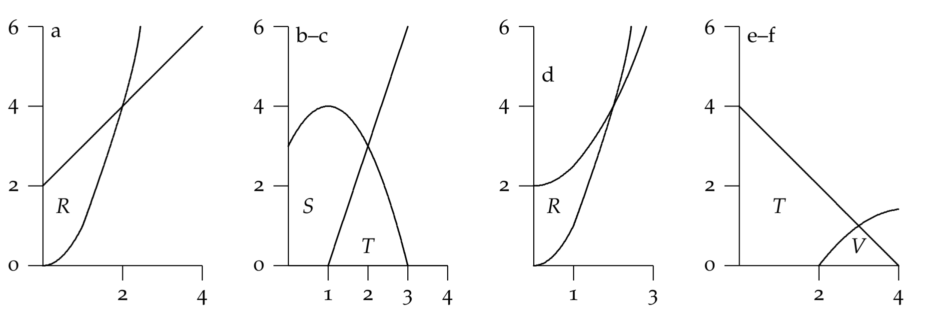


Figure 6.33: The regions for YOU TRY IT 6.29 .

**YOU TRY IT 6.30 (Do by shells).** Let $R$ be the region in the first quadrant enclosed by $y = \sqrt{9-x^2}$ and the two axes. Find the volume generated by rotating $R$ about the $y$-axis.

**YOU TRY IT 6.31 (Do by shells).** A small canal bouy is formed by taking the region in the first quadrant bounded by the $y$-axis, the parabola $y = 2x^2$, and the line $y = 5 - 3x$ and rotating it about the $y$-axis. (Units are feet.) Find the volume of this bouy. Compare to YOU TRY IT 6.16 .

**YOU TRY IT 6.32.** These two problems are rotations about the $y$-axis, so either disks or shells are possible. **However**, in each case only one of these methods is easy. That's why it is important to know both.

($a$) Let $R$ be the region in the *right half-plane* enclosed by $y = 4 - x^2$ and $y = x^4 - 4x^2$. Rotate this region about the $y$-axis and find the volume. Disks or shells: Only one method is possible. (Answer: $56\pi/3$.)

($b$) Let $R$ be the region in the first quadrant enclosed by $y = \ln x$, $y = 0$, and $x = e$. Rotate the region about the $y$-axis and find the resulting volume. (Answer: $\pi(e^2+1)/2$.)

($c$) **Extra Fun.** Rotate this region about the $x$-axis and find the resulting volume. (Answer: $\pi(e-2)$.)

**YOU TRY IT 6.33 (Extra Credit.).** Try these.

($a$) Let $y = \frac{1}{x}$ on $[1, a]$, where $a > 1$. Let $R$ be the region under the curve over this interval. Rotate $R$ about the $x$-axis. What value of $a$ gives a volume of $\pi/2$?

($b$) What happens to the volume if $a \to \infty$? Does the volume get infinitely large? Use limits to answer the question.

($c$) Instead rotate $R$ about the $y$-axis. What value of $a$ gives a volume of $\pi/2$? This is particularly easy to do by shells!

($d$) Instead rotate $R$ about the line $y = -1$. If we let $a = e$ what is the resulting volume?

**YOU TRY IT 6.34 (Which is it?).** Let $R$ be the region enclosed by $y = 2x^2$ and $y = x^4 - 2x^2$ in the *right half-plane* where $x \geq 0$.

($a$) Find the *area* of $R$.

($b$) Rotate $R$ around the $y$-axis and find the *volume* generated. Shells or disks? Only one method is possible. (Answer: $32\pi/3$)