Hashtables

Balanced search trees provide O(log n) find, insert, remove.
But can we do better?

O(1) would be the logical goal to strive for.
But how?

Observations.

- find is presumably the most commonly-used operation for Map, so it should be most efficient
- arrays have O(1) lookup by index

So – can we find a way to convert a key to an integer array index in O(1) time?

Hashtables

Let N be the size of the array.

- key → index is easy if the key is already an integer 0..N-1

Otherwise use a *hash function* h(k) to convert key k to an index.

- e.g. $h(k) = k \bmod N$ if k is an integer
- e.g. $h(k) = \sum a^{|k|-(i+1)} \text{char}(k_i) \bmod N$ if k is a string
  - a = size of the alphabet
  - char(c) maps c to an integer 0..a-1

Hash Functions

Challenges.

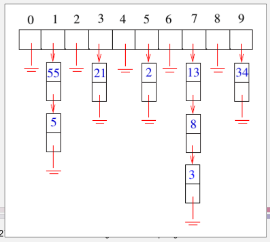
- h(k) must be efficient to compute, since it must be computed for every find, insert, remove operation
  - $h(k) = k \bmod N$ → O(1)
  - $h(k) = \sum a^{|k|-(i+1)} \text{char}(k_i) \bmod N$ → O(|k|)

  Must factor in this time if not O(1) – though it often depends on something which is in practice a constant with respect to *n*.
- h(k) typically maps a large range of key values into the much smaller range 0..N-1 so collisions may occur
  - should spread keys over indexes as evenly as possible
    - choosing N to be a reasonably large prime helps with this
      - (but there is a tradeoff – larger N means more space for hashtable)
  - sensitive to particular distribution of keys in a given application

Collision Resolution

What to do with two elements whose keys hash to the same value?

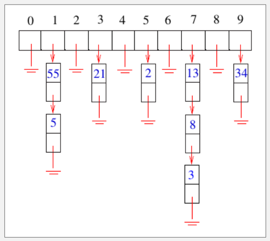
- separate chaining – store a list of elements at each slot in the array
- open addressing – find an alternate slot if the desired one is full



0	1	2	3	4	5	6	7	8	9
34	5	55	21		2		3	8	13

Separate Chaining

- operations
  - find – compute h(k), then search that list for desired key
  - insert – compute h(k), then add to that list
  - remove – find + remove from list



Separate Chaining

Perform the following operations on an initially empty hashtable of size 7 using chaining and the hash function h(x) = x%7. Show the contents of the hashtable after each step, putting the head of the list first and separating

insert 35	delete 10
insert 10	delete 24
insert 18	delete 11
insert 24	insert 74
insert 5	
insert 11	

Separate Chaining

- expected size of each list is n/N
  - assuming hash function distributes keys well
  - reduces to O(1) if n ≤ N or is never more than a fixed multiple of N i.e. hashtable is not too full
- typical implementations use unsorted linked lists
  - insert – O(1)
  - find, remove
    - expected O(n/N) if keys are well distributed
      - reduces to O(1) if n/N is bounded (e.g. n < N)
    - worst case O(n) if all keys hash to same index
  - can add move-to-front heuristic if some keys are searched for more frequently than others
  - overhead for storing pointers

Separate Chaining

- what about sorted linked lists?
  - can't exploit binary search with linked lists, but approximately halves the cost of an unsuccessful search for find, remove
  - insert O(n/N)
- what about arrays?
  - find is faster if sorted (binary search) but then have cost of shifting on insert/remove
  - still have space overhead (empty slots to avoid frequent shrinking/growing) + time overhead (shrinking/growing)

Separate Chaining

- more sophisticated implementations – array-based
  - eliminate space overhead – use an array of size k for a list of k elements (*dynamic array*)
    - no linked list pointers or empty slots
    - can exploit hardware features that provide greater efficiency for dealing with sequential memory positions
    - adds cost of array resizing on insert, remove
  - eliminate search through chain – use a hashtable of size k^2 for a list of k elements with a perfect hash function (no collisions), rebuilding when a collision occurs (*dynamic perfect hashing*)
    - guaranteed O(1) worst-case find
    - low amortized insert time – rebuilding is infrequent because load factor of secondary tables is 1/k
    - with N = O(n), expected total space is O(n), worst case $O(n^2)$

Separate Chaining

- more sophisticated implementations – other data structures
  - O(log n) operations – balanced search tree
    - O(log n) worst case for find, insert, remove
    - additional overhead not generally worth it except in special cases
      - e.g. high load factor (n/N ≥ 10)
      - e.g. likely non-uniform hash distribution (some long chains)
      - e.g. need to guarantee good performance in worst case
    - using a larger hash table or finding a better hash function may be better alternatives

Open Addressing

- requires n ≤ N

If h(k) is full, follow a *probe sequence* to locate element / find first empty slot for insertion.

- linear probing – h(k) + c·i [c is often 1]
  - c should be relatively prime to N (not a problem if N is prime)
  - *sequential probing* when c=1
- quadratic probing – $h(k) + i^2$
- double hashing – h(k) + i h'(k)

Open Addressing

Perform the following operations on an initially empty hashtable of size 7 using open addressing with sequential probing and the hash function h(x) = x%7. Show the contents of the hashtable after each step. Use a - (a dash) to

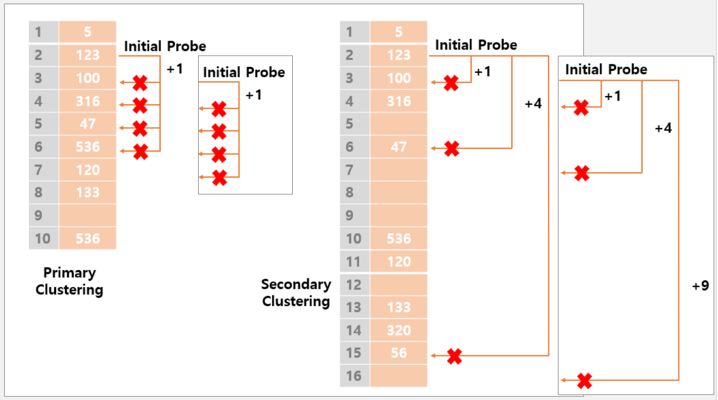
insert 35
insert 10
insert 18
insert 24
insert 5
insert 11

– linear probing – h(k) + c·i [c is often 1]
c should be relatively prime to N (not a problem if N is prime)
sequential probing when c=1

– quadratic probing – $h(k) + i^2$

– double hashing – h(k) + i h'(k)

Primary and Secondary Clustering



Open Addressing

- linear probing – h(k) + c·i [c is often 1]
  - exhibits better memory locality than other options
  - suffers from clustering
    - keys that hash to the same index or adjacent indexes interfere with each other
    - performance degrades quickly as n approaches N
  - sensitive to key distribution
    - uneven key distribution exacerbates the clustering problem
- quadratic probing – h(k) + i^2
  - suffers from secondary clustering
    - keys that hash to adjacent slots have adjacent probe sequences
  - may not find an empty slot even if one exists
- double hashing – h(k) + i h'(k)
  - expected length of unsuccessful probe sequence is 1/(1-α) → O(1) if table is not too full
    - α = n/N (load factor)

Open Addressing

Deletion requires special handling.

- can re-insert all elements following the deleted element
  - if the load factor is low enough, this should only be a small number of elements
  - limited to sequential probing

delete 24

delete 10

delete 35

delete 18

0	1	2	3	4	5	6
35	11	-	10	18	24	5

Open Addressing

Deletion requires special handling.

- can mark empty slot as "deleted" – find continues on, insert can fill
  - drawback: probe sequence lengths are based on the largest the collection has been, not the current size
  - solution: can periodically re-hash everything to clean up

delete 24

delete 10

delete 35

delete 18

0	1	2	3	4	5	6
35	11	-	10	18	24	5

Hashtables

If done properly, hashtables provide O(1) expected time for find, insert, remove – once h(k) has been computed.

- "done properly" means load factor isn't too high and is kept bounded, and there is good distribution of hash values

Computing h(k) can take time.

- e.g. for strings, computing h(k) = O(|k|) … which reduces to O(1) if |k| is bounded, but must be considered as O(|k|) otherwise

Worst-case behavior is O(n) for find and remove, unless separate chaining + a fancier bucket implementation is used (which has memory overhead).

- worst case occurs when key distribution is poor and load factor is high

Hashtables

What about other operations?

- initialization
  - O(N) – size of the array used for the hashtable
- traversal
  - in most cases O(n+N) for separate chaining – must examine each index of table as well as all elements
    - can be worse e.g. worst case dynamic perfect hashing
  - O(N) for open addressing
- find next larger/smaller key, find min/max key
  - full traversal is required because h(k) does not preserve original ordering of keys

Map/Dictionary Implementation Recap

Dictionary operation	Unsorted array	Sorted array	Singly linked unsorted	Singly linked sorted	balanced BST	hashtable
Search(A, k)	$O(n)$	$O(\log n)$	$O(n)$	$O(n)$	O(log n)	O(1) expected
Insert(A, x)	$O(1)$	$O(n)$	$O(1)$	$O(n)$	O(log n)	O(1) expected
Delete(A, x) or Delete(A,k) (given location of *x*)	$O(1)^*$	$O(n)$	O(1) *	O(1) *	O(log n)	n/a
Remove(A,x) or Remove(A,k) (not given location of *x*)	O(n) requires search + delete	O(n)	O(n)	O(n)	O(log n)	O(1) expected

OrderedDictionary

Ordered Dictionary operation						
Search(A, k)						
Insert(A, x)						
Delete(A, x)						
Successor(A, x)						
Predecessor(A, x)						
Minimum(A)						
Maximum(A)						