You're planning to drive from Boston to Seattle on I-90 this summer, and have a GPS programmed with the locations of gas stations along the way. Assuming that your car can go 400 miles on a tank of gas, determine where you should stop for gas in order to make as few stops as possible.

Establish the problem.

- specifications – task, input, output, legal solution, optimal solution

Task:

You're planning to drive from Boston to Seattle on I-90 this summer, and have a GPS programmed with the locations of gas stations along the way. Assuming that your car can go 400 miles on a tank of gas, determine where you should stop for gas in order to make as few stops as possible.

Input: locations of *n* gas stations

Output: which gas stations to stop at

Legal: no consecutive gas stations are more than 400 miles apart

Optimal: fewest stops

- examples

Identify avenues of attack.

- targets
- approach

Series of choices.

- paradigms, patterns, and flavors

Paradigm: greedy.

Flavor: subset

Pattern: produce output – which gas station to stop at next

process input – for the current gas, stop or not?

- greedy choices and counterexamples

series of choices: the next gas station to stop at

greedy choices:

we have: locations of gas stations, (can compute) how far from our current position

greedy choice: next station: farthest (towards Seattle) <= 400 miles away

Define the algorithm.

- main steps – repeat make the next choice until done

repeat

the next stop is: farthest (towards Seattle) station <= 400 miles away

until we are within 400 miles of Seattle

- exit condition

when we are within 400 miles of Seattle

- setup

current location is Boston

assumption: full tank of gas

- wrapup

(for the route) drive the rest of the way to Seattle

- special cases

what if there's a gap of more than 400 miles between consecutive gas stations (in the input)?

- algorithm

pre: at least one gas station

current location is Boston, tank is full (we assume)

repeat

  the next stop is: farthest (towards Seattle) station <= 400 miles away (also > 0 miles away)

  if there isn't one, exit (no solution)

until we are within 400 miles of Seattle

Show termination and correctness.

- termination
    - measure of progress

distance left to Seattle

    - making progress

next gas station is > 0 miles closer to Seattle

    - the end is reached

we keep decreasing the distance to Seattle, eventually be within 400 miles

- correctness
    - loop invariant

after the first k stops (in order from Boston to Seattle) –

- no stops are more than 400 miles apart
- the algorithm's kth stop is at least as far from Boston as the optimal's kth stop
    - establish the loop invariant

k = 1

- no stops are more than 400 miles apart because there's only one stop (and the first stop picked is at most 400 miles from Boston)
- the algorithm's 1st stop is at least as far from Boston as the optimal's 1st stop because the algorithm picked the farthest stop from Boston <= 400 miles – if the optimal's first stop is farther, it's not reachable on one tank of gas

- maintain the loop invariant

assume: after the first k stops (in order from Boston to Seattle) –

- no stops are more than 400 miles apart
- the algorithm’s kth stop is at least as far from Boston as the optimal’s kth stop

show: after the first k+1 stops (in order from Boston to Seattle) –

- no stops are more than 400 miles apart
- the algorithm’s (k+1)th stop is at least as far from Boston as the optimal’s (k+1)th stop

legal: alg doesn’t pick a stop more than 400 miles from last

optimality: by contradiction – assume this is where the alg screws up: after k+1 stops the algorithm’s (k+1)th stop is less far from Boston than the optimal’s (k+1)th stop

idea: opt’s k+1 stop <= 400 miles from opt’s k stop, but opt’s k stop <= alg’s in terms of dist from boston, so opt’s k+1 stop <= 400 miles from alg’s k stop so legal for alg to pick and alg would have picked

- final answer

loop invariant: after the first k stops (in order from Boston to Seattle) –

- no stops are more than 400 miles apart
- the algorithm’s kth stop is at least as far from Boston as the optimal’s kth stop

exit condition: we are within 400 miles of seattle (k stops is all the stops we’re going to make)

show: k is the fewest number of stops possible

consider the cases: |A| < |O| or |A| > |O| or |A| = |O|

- |A| < |O| – impossible – O isn’t optimal in that case
- |A| > |O| - if the alg picks more stops, then at |O| stops it was > 400 miles from seattle – but then the opt was too because alg’s |O|th stop >= opt’s |O|th stop
- the only thing left: |A| = |O|

Determine efficiency.

- implementation
- time and space
- room for improvement