Math 110 Graph Theory II: Circuits and Paths

For Next Time. Read Section 6.1 “Circuit Training” (p. 386ff) for more background on this material. Review the definition of a **graph**. Make sure you understand these terms on the following pages: **path**, **circuit**, **connected**, and **degree**.

Three Problems

Today you will spend part of class trying to solve these two problems. One should be familiar. I will not be giving many hints. Most mathematical problems are like these two. They are very open-ended. You don’t know how to find the solution or even whether there is a solution! That’s what makes mathematics fun. It’s like a puzzle and your innate curiosity should motivate you to try to find a solution. We will break after 10–15 minutes to see what progress has been made and then work on them some more.

The Handshake problem. A party consists of three couples. Alice and Ben are the hosts. Several of the people shake hands when they meet following these reasonable restrictions:

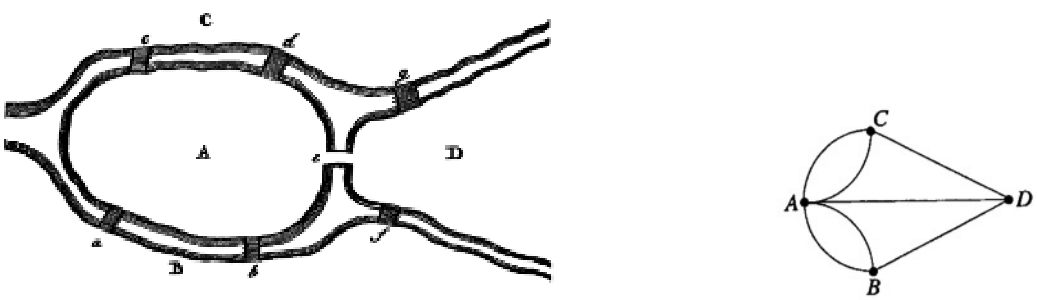
a) No two people shake hands with each other more than once.

b) No one shakes hands with the person with whom they came to the party.

After the handshaking is complete, Alice asks everyone else how many hands each person shook. She gets a different answer from each person. How many hands did Alice shake? How manny did Ben shake? (Remember, Ben is her partner.)

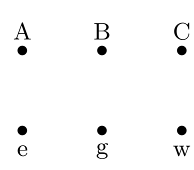
Work in groups to solve this problem. My only hint is this: Make a diagram (graph!) that corresponds to the six people. Label them A(lice) and B(en) who are one couple, C(asey) and D(on) the second couple, and E(llen) and F(rank) the last couple. Try working out the solution.

The Königsberg bridge problem. The city of Königsberg (below) was formerly in Germany but is now known as Kaliningrad and is part of Russia. The river Preger runs through the city and in the 18th century there were seven bridges over the river. This problem asks if the seven bridges can all be traversed in a single trip without doubling back over any bridge, with the additional requirement that the trip ends in the same place it began. The bridges are denoted by lowercase letters, ‘a’ through ‘g’. The uppercase letters represent land. (By the way this problem was first proposed and solved by Euler, the person after whom the Euler characteristic is named.)



The Königsberg bridges

The Utility problem. The Adamses, Bushes, and Clintons each have built houses that need to be connected to three utilities: electric, gas, and water. Suppose all utility lines must run on the ground and no two can intersect. How can this be done? Draw and label graph. You may wish to rearrange (move around) the vertices.



Graph Theory Dictionary

To be able to use graph theory to analyze more interesting and complex problems, there is some basic vocabulary that you need to master. Make sure you can define each of the following terms and give an example of each.

1. **Graph:** A **graph** is a finite non-empty set of **vertices** (dots) along with a set of **edges** (segments, arcs) between pairs of vertices. If we represent vertices by letters A, B, C, ..., then edges can be represented as pairs of letters, e.g., AC, BC, AA, ... as appropriate.

2. A **loop** is an edge from a vertex back to the same vertex. E.g., AA or BB.

3. **Path:**

4. **Circuit:**

5. **Euler Path:**

6. **Euler Circuit:**

7. **Connected Graph:**

8. **Degree of a vertex:**

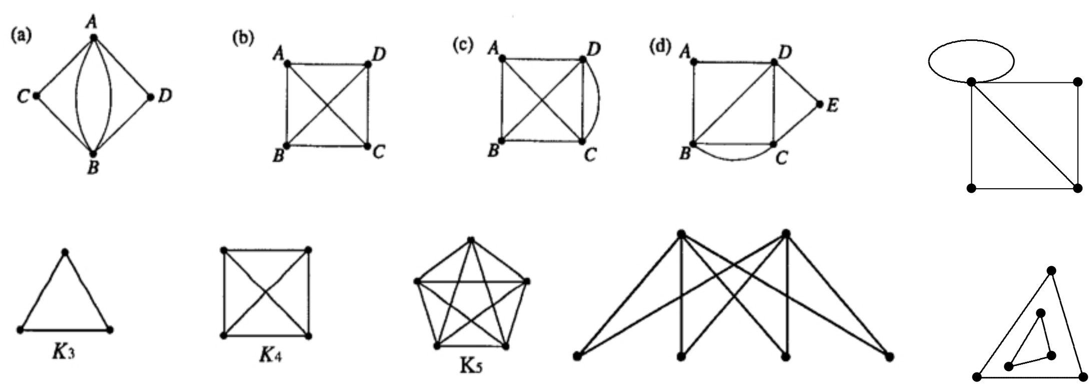
Mindscape 1: Section detection. Graphs can be used to represent lots of relations or networks. For example, suppose that we let the vertices represent students in Math 110. We can draw an edge between student A and B if they are in the *same* section of the course. What would a loop represent? Here are 7 students in Math 110 this term: AJ, Brett, Cormac, Demi, Eli, Georgia, and Max. Draw the graph (the edges may cross) for this situation. Figure out who's in which section first. Is the graph connected?

Mindscape 2: a, e, i, o, u. Draw another graph above for the same set of students using edges to indicate that names that both begin with vowels or both with consonants.

Definition: The **degree** of vertex is the number of edges meeting at the vertex. Note that a loop adds two to the degree of a vertex since it touches it twice. Label the degrees of the vertices in the Köningsberg graph on page 1.

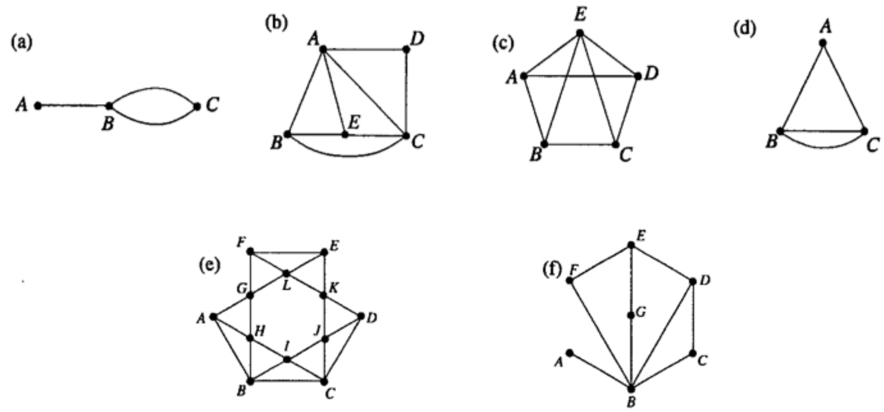
Mindscape 3: Count on me. Label each vertex in your section graph with its degree. Label each vertex in the graphs in Mindscape 4 with its degree. Check with your partner.

Mindscape 4. Consider the graphs below. Determine which have an **Euler circuit**. If it does, show your answer by labeling the edges 1, 2, 3, etc.



Mindscape 5: No can do. State a general rule for when a connected graph G **cannot** have an Euler circuit. The term degree should appear in your rule. Give a *justification* for this rule.

Mindscape 6. Even if there is not an Euler circuit, there may still be an **Euler path**. Determine which of the following graphs have an Euler path. (Label 1, 2, 3, etc.) Try one more of your own. Label the degrees of each of the vertices.



Mindscape 7. No can do, redux. State a general rule for when a connected graph G **cannot** have an Euler path. Give a justification.

Mindscape 8. Look back at the graphs in Mindscape 6. Count the number of edges in each and record the information below. Then add up the degrees of all the vertices in each graph. What pattern do you find? Try one more graph of your own. State your pattern as a theorem:

The sum of the degrees of all the vertices in a graph ____________________________________.
Clearly explain why this happens.

Graph	**a**	**b**	**c**	**d**	**e**	**f**	Your own
Edges							
Degree sum							

Mindscape 9. It's a matter of degree.

a) A graph has seven vertices. Six vertices have degrees 1, 3, 3, 4, 4, and 6, respectively. If there are 12 edges in the graph what is the degree of the missing vertex? Carefully explain your reasoning.

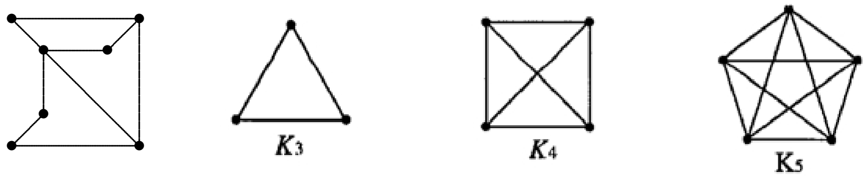
b) A graph has 9 vertices. Eight vertices have degrees 1, 2, 3, 3, 3, 4, 4, and 6, respectively. Is the degree of the missing vertex odd or even or could it be eiether? Carefully explain your reasoning.

c) Can a graph have just one odd degree vertex? Explain.

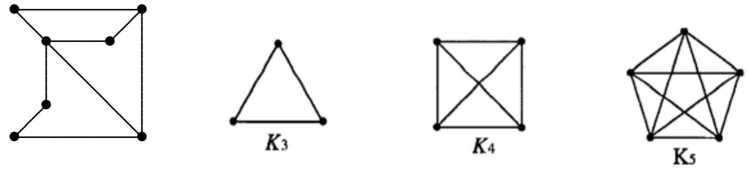
Math 110 Graph Theory II: Group Hand In. Names: ______________________

Mindscape A: Circuit Training.

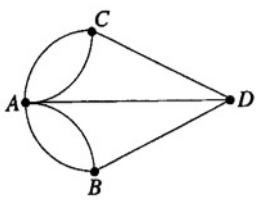
a) Which graphs below, if any, have an Euler Circuit? Draw any such circuit by numbering the edges.



b) Which graphs below, if any, have an Euler Path (but not a circuit)? Draw any such path by numbering the edges.



Mindscape B: The Königsberg bridge problem. The river Preger runs through tcity of Königsberg and in the 18th century there were seven bridges over the river. Can all seven bridges (edges) be traversed in a single trip without doubling back over any bridge, with the additional requirement that the trip ends in the same place it began?



a) Circle the correct answer. What is the problem asking you to find: a path, a circuit? an Euler path, or an Euler circuit?

b) Using the theorems we have developed, answer the question in one or two sentences.

Mindscape B: The fifth degree.

a) A graph has 5 vertices. The first 4 have degrees 1, 2, 3, and 6, respectively. If there are 8 edges in the graph what is the degree of the missing vertex? Explain your reasoning.

b) A graph has 3 vertices. Can the degree of each vertex be odd? Either draw an example or if it is impossible explain briefly.