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# Math 110: An Introduction to the Mathematics of Voting

## 1.1 What's The Problem?

Voting theory is the mathematical description of the process by which democratic societies resolve the different and conflicting views of the group's members into a single choice for the group. Each vote is an expression of that voter's preference about the outcome of an election. Why do we need a mathematical theory about something so simple as voting? How hard is it be to find a simple, fair, and consistent procedure for determining the outcome of an election?

In the United States with its two party system we have become used to elections that involve only two major candidates. Suppose that there is an election between Professors Critchlow and Eck for chairmanship of the Department of Mathematics and Computer Science. In this case the situation is as simple as you might imagine. How should the winner be determined?[1]

However, many elections involve making a decision from among more than two candidates or choices. For example, think about how might this class decide whether to have the second exam on November 5(W), 7(F), or 10(M)?

In my lifetime, at least two presidential elections have been greatly affected by third party candidates. In 1968 Richard Nixon (R) received 43.4% of the vote, Hubert Humphrey (D) received 42.7%, and George Wallace (American Independent Party) received 13.5%. If Wallace had not run the results could easily have been different.

More recently in the 2000 election Gore (D) received 48.38%, Bush (R) 47.87% and Nader (Green Party) 2.74% and Bush won. (Why?) The perception at the time was that most of those who voted for Nader would have voted for Gore if Nader had not run. (Why?) This likely would have changed the outcome of the election. For similar thoughts about this year's election, at least how it was viewed in late July, see the article entitled "The Power of the Protest Vote" at `http://campaignstops.blogs.nytimes.com/2008/07/29/the-power-of-the-protest-vote/index.html` in *The New York Times*.

In the material that follows we consider various attempts to decide an election *fairly* when there are three or more choices.

[1]You probably said that the candidate with the majority of the votes should be elected. (Recall that a **majority** means greater than 50% (half) of the votes.)



## 1.2 The Plurality Method

The simplest and perhaps most common (?) method of deciding an election between candidates is the familiar plurality method.

**DEFINITION 1.1** (Plurality Method). In the **plurality method**, each voter selects one candidate on the ballot. The winner is the candidate with the most votes. Note that the winner does not need to have a majority of the votes. (In the rare case of a tie, some other method must be used to select the winner. We will largely ignore the possibility of ties in what follows.)

**EXAMPLE 1.1.** In a 3 candidate election, candidate A gets 12 votes, candidate B gets 20 votes, and candidate C 18 gets votes (total of 50 voters). Candidate B has a plurality of the votes but, in this case, not a majority of the votes. Nonetheless, Candidate B is the winner.

Assuming that there are no ties, the plurality method is easy to carry out. However there are problems with this method when there are more than two candidates, especially when two of the candidates are similar and split what potentially would be a majority of the vote. The next section examines this more closely.

☞ **Mindscape 1.1.** Note: You should try these exercises as you encounter them in the reading. This particular exercise is the one place where ties are considered.

(*a*) Devise a fair method to determine the winner of a three-way election when the two candidates with the most votes are tied.

(*b*) Now devise a method to determine the winner of a three-way election when all three candidates receive the same number of votes.

## 1.3 Vote-for-Two

Another simple voting method in elections where there are *more than two candidates* is **vote-for-two-voting**. Each voter must vote for two different candidates and the candidate with the most votes wins. The idea is that this method should elect a candidate that is acceptable to the most people.

**EXAMPLE 1.2.** The 1992 presidential election pitted three candidates against each other: Bill Clinton, George Bush, and Ross Perot. Suppose that voters had been asked to vote-for-two candidates on the ballot with the following outcome.

**TABLE 1.1.** Vote-for-two candidates for the 1992 U.S. Presidential Election

| | 36 Voters | 8 Voters | 30 Voters | 9 Voters | 7 Voters | 13 Voters |
|---|---|---|---|---|---|---|
| Clinton | ✓ | ✓ | | ✓ | ✓ | |
| Bush | | ✓ | ✓ | ✓ | | ✓ |
| Perot | ✓ | | ✓ | | ✓ | ✓ |

In Table 1.1 The numbers across the top indicate how many voters voted for the pair of candidates checked in the column below. The first column in Table 1.1 indicates that 36 voters checked off

Clinton and Perot. Looking across all columns, we see that Clinton received $36+8+9+7=60$ votes. Bush received $8+30+9+1=48$ votes. Perot received $36+30+7+13=86$. Thus, using the vote-for-two method, Perot was easily the winner. (If the totals across the top represented millions of votes, this would be the approximate result of the 1992 election.) We will see why Clinton actually won the election in the next example.

## 1.4 Preference Rankings

There are two fundamentally different types of voting methods. The methods differ in whether or not they ask a voter to state a preference between two given alternatives. Those systems which do are called **preferential voting methods**. The vote-for-two method is an example of a non-preferential or so-called **approval voting method** and they will be discussed in more detail later.

The plurality method described in Example 1.1 is an example of an approval method. Each voter picks one alternative, presumably that voter's first choice, and whichever alternative has the most votes wins.

However, in most voting situations, each voter has an order of preference of the candidates. This ordering is called a **preference ranking**. The opinions of a given voter can be thought of as a preference ballot, which lists each alternative in order. With this model, we see that the plurality method considers only the first-place rankings, ignoring the rest of information in preference ballot.

**EXAMPLE 1.3.** Return to the 1992 presidential election pitted that pitted Bill Clinton, George Bush, and Ross Perot against each other. Suppose that voters had been asked to rank the candidates on the ballot (as they do in Australia) with the following outcome.

**TABLE 1.2.** Preference rankings for the 1992 U.S. Presidential Election

| | 36 Voters | 8 Voters | 30 Voters | 9 Voters | 7 Voters | 13 Voters |
|---|---|---|---|---|---|---|
| Clinton | 1 | 1 | 3 | 2 | 2 | 3 |
| Bush | 3 | 2 | 1 | 1 | 3 | 2 |
| Perot | 2 | 3 | 2 | 3 | 1 | 1 |

In Table 1.2 the numbers across the top indicate how many voters ranked the candidates in a particular preference pattern. The numbers in the interior of the table indicate the rank of the candidate. The first column in Table 1.2 indicates that 36 voters ranked Clinton first, Bush third, and Perot second. Looking across all columns, we see that Clinton received $36+8=44$ first-place votes. Bush received $30+9=39$ first-place votes. Perot received $7+13=20$ first-place votes. Thus, using the plurality method, Clinton was the winner. (If the totals across the top represented millions of votes, this would be the approximate result of the 1992 election.)

This example is related to Example 1.2 in the following way. As might be expected, the ✓'s in Table 1.1 correspond to rankings of 1 or 2 in Table 1.2. After all, in a vote-for-two election it makes sense for voters to actually vote for their top two ranked candidates. So how do we explain that Perot was a clear winner using the vote-for-two method but a distant third using the plurality method? One possible explanation is since Clinton and Bush supporters were Democrats and Republicans, respectively, while Perot supporters were independents. Neither Democrats nor Republicans were very willing to vote for the other party when forced to vote for two candidates and, hence, Perot was a frequent 'second choice.' But the vote-for-two method does not distinguish between first and second choices when votes are counted—both count the same. So Perot's election would have been the result of a large group of voters selecting him as their second choice.



So which method do you think is better or fairer? In this situation vote-for-two elects a person acceptable to a large majority of voters (86 out of 103) while plurality elects Clinton with a minority of votes (44 of 103). The answer is not easy and each society must choose how to elect its representatives. What we will see is that no method is perfect—each method has its flaws.

#### Vote Splitting

The next example illustrates what can happen when there are two candidates with very similar positions with which the majority voters agree and a third candidate with different views and the committed support of a minority of voters.

**EXAMPLE 1.4.** Three candidates are running for the position of Chair of the Board of Trustees of HWS: Smith, Smythe, and Jones. Smith and Smythe have positions on the issues (new buildings, financial aid, capital campaign) that are very similar. Those of Jones are different, e.g., no financial aid to save money. The 15 Board members vote with the outcome in Table 1.3.

**TABLE 1.3.** Preference rankings for Chair of Board of Trustees of HWS

| | 5 Voters | 4 Voters | 3 Voters | 3 Voters |
|---|---|---|---|---|
| Smith | 1 | 2 | 3 | 2 |
| Smythe | 2 | 1 | 2 | 3 |
| Jones | 3 | 3 | 1 | 1 |

Notice that Jones is a polarizing candidate: Voters either love her (rank her first) or dislike her (rank her last). Smith and Smythe are nearly indistinguishable both in their positions and voter support. If the plurality method is used, then Jones wins with 6 votes to Smith's 5 and Smythe's 4. Smith and Smythe have split the majority vote (9 of 15 or 60%) and HWS has ended up with a chair that only a minority wanted and that the majority find highly undesirable (9 of 15 voters rank Jones last).

Vote splitting of the sort in the previous example is a major fault of plurality voting. The outcome of elections in such situations seems to have distorted the will of a substantial majority of voters. There have been a number of methods proposed to deal with such issues.

☞ **Mindscape 1.2.** Is there a method that better captures the views of the voters? Specifically, looking at the entire set of preference rankings, which candidate do you believe should have been elected Chair? Why?

There are several ways of taking the additional information expressed in the preference rankings into account that might better capture the sentiments of the electorate and some of these are explored in the next few sections. See if your method is discussed.

## 1.5 Plurality with Runoff

Throughout the remainder of this unit we will make the following reasonable assumptions about preference ballots.

**Preference Assumptions**

1. If a voter ranks one candidate higher than another, then if the voter had to choose between only those two candidates, the voter would choose the higher-ranked one.
2. The order of preference is not changed if one or more of the candidates is eliminated, as in a runoff election.

One method of selecting a winner when no candidate receives a majority of the votes is to have a run-off election. There are several ways to do this.

**DEFINITION 1.2** (Plurality With Instant Runoff)**.** Assume that preference ballots have been used and no candidate has received a majority of first-place votes. One runoff method is to pit the two candidates who received the most first-place votes against each other. Then using the preference assumptions listed above, the runoff winner can be determined immediately without a second vote (assuming there is no tie). Because this process can be done without revoting, it is sometimes called **plurality with instant runoff**.

**EXAMPLE 1.5.** Return to Example 1.3. None of the presidential candidates received a majority of the votes. Clinton and Bush were the two candidates receiving the greatest number of first-place votes, 44 and 39, respectively. In a runoff election, the preference assumptions above mean that those who preferred Clinton to Bush will continue to do so, and vice versa, and would vote accordingly. Since we already know their preferences, there is no need for a second vote, rather we just need to examine the ranking information.

The 44 voters who ranked Clinton first will continue to do so. Look at the last two columns of Table 1.2. Notice that 7 of Perot's voters prefer Clinton (ranked 2) to Bush (ranked 3). So Clinton now has $44+7 = 51$ votes On the other hand, 13 of Perot's voters prefer Bush (ranked 2) to Clinton (ranked 3). So Bush ends up with $39 + 13 = 52$ votes in the runoff election. Consequently in this (imaginary) runoff election, Bush would have beaten Clinton in the 1992 presidential election. We now have examined three different (reasonable) voting methods which produce three different winners for the 1992 presidential election!

☞ **Mindscape 1.3.** Take time right now to think about this result. Without letting your own political interests influence your decision, which of the two election methods, plurality or plurality with runoff, better reflect the views of the voters? Justify your reasoning. Where does vote-for-two rank compared to these two plurality methods?

☞ **Mindscape 1.4.** Return to Example 1.4 and Table 1.3. None of the HWS candidates received a majority of the votes. Determine the plurality with runoff winner. Does this candidate seem to better embody the views of the Board compared to the simple plurality winner?

☞ **Mindscape 1.5.** Devise your own runoff procedure different from 'plurality with runoff' for an election with four candidates that makes use of the preference rankings if there is no majority winner.



## 1.6 Borda's Method

Elections determined by the plurality method or the plurality method with runoff do not take into account voters relative preferences for all of the candidates. Even when a runoff is used, this takes into account the relative preference of just two candidates. The following election method uses all of the ranking information on a voter's ballot.

**DEFINITION 1.3** (The Borda Count Method)**.** For the Borda Count Method, each candidate (or alternative) gets 1 point for each first-place vote received, 2 points for each second-place vote, etc., all the way up to $N$ points for each last-place vote (where $N$ is the number of candidates/alternatives). The candidate with the *smallest* point total is the **Borda winner** the election.

Note that smaller point totals are better (like golf) since they indicate higher rankings. The Borda Count Method, or some variation of it, is often used for polls such as those which rank sporting teams or academic institutions.

See http://www.simnet.is/bss/English/SkjolEnsk/votes08.htm for some interesting historical comments on the Borda Method.

**EXAMPLE 1.6.** Return yet again to the imaginary preference rankings in the 1992 presidential election. Determine the Borda count for each of the candidates and the Borda winner.

| | 36 Voters | 8 Voters | 30 Voters | 9 Voters | 7 Voters | 13 Voters |
|---|---|---|---|---|---|---|
| Clinton | 1 | 1 | 3 | 2 | 2 | 3 |
| Bush | 3 | 2 | 1 | 1 | 3 | 2 |
| Perot | 2 | 3 | 2 | 3 | 1 | 1 |

**SOLUTION.** Here $N = 3$ is the number of choices. The Borda count for Clinton is given by:

$$\begin{aligned}\text{(first-place votes)} \times 1 + \text{(second-place votes)} \times 2 + \text{(third-place votes)} \times 3 \\ &= (36 + 8) \times 1 + (9 + 7) \times 2 + (30 + 13) \times 3 \\ &= 44 + 32 + 129 \\ &= 205.\end{aligned}$$

The Borda count for Bush is

$$(30 + 9) \times 1 + (8 + 13) \times 2 + (36 + 7) \times 3 = 39 + 42 + 129 = 210.$$

The Borda count for Perot is

$$(7 + 13) \times 1 + (36 + 30) \times 2 + (8 + 9) \times 3 = 20 + 132 + 51 = 203.$$

The Borda winner is Perot by a nose.[2] Much of Perot's appeal was that he was an alternative to either of the mainstream candidates. Though the overwhelming majority did not prefer him to both of the two others, a substantial number did prefer him to at least one of the candidates. Hence, he conceivably could have won the election using the Borda method. The fact that in this imaginary preference election a large number ranked Perot second while the other two candidates were polarizing (generally ranked either first or last by voters) made it possible for Perot to 'win.'

[2] A bad joke since most cartoonists drew Perot with a large nose and large ears. A particularly interesting cartoon appears at http://www.greenberg-art.com/.toons/.Toons,%20favorites/The_alternative.html

☞ **Mindscape 1.6.** Notice that the three preference voting methods we have described have produced three different winners for the 1992 presidential election with three candidates. Which method was *fairest*? Support your answer.

☞ **Mindscape 1.7.** Suppose that in a survey 100 people are asked to rank their ice cream preferences with the results given below.

| | 33 Voters | 3 Voters | 10 Voters | 20 Voters | 7 Voters | 27 Voters |
|---|---|---|---|---|---|---|
| Chocolate | 1 | 1 | 2 | 3 | 2 | 3 |
| Vanilla | 2 | 3 | 1 | 1 | 3 | 2 |
| Mocha | 3 | 2 | 3 | 2 | 1 | 1 |

Determine the Borda winner, the plurality winner, and the plurality winner with instant runoff between the first and second place finishers in the plurality election.

## 1.7 Condorcet Winners: Head-to-Head Comparisons

Another method of determining the winner of an election when we know the preference rankings of each voter involves pitting each candidate against every other candidate in a series of head-to-head comparisons.

**EXAMPLE 1.7.** Return a fifth time to the imaginary preference rankings in the 1992 presidential election.

| | 36 Voters | 8 Voters | 30 Voters | 9 Voters | 7 Voters | 13 Voters |
|---|---|---|---|---|---|---|
| Clinton | 1 | 1 | 3 | 2 | 2 | 3 |
| Bush | 3 | 2 | 1 | 1 | 3 | 2 |
| Perot | 2 | 3 | 2 | 3 | 1 | 1 |

(*a*) Who would be the winner in a head-to-head comparison between Clinton and Bush?

(*b*) Who would be the winner in a head-to-head comparison between Clinton and Perot?

(*c*) Who would be the winner in a head-to-head comparison between Bush and Perot?

**SOLUTION.** In a head-to-head comparison voters are asked to compare just two candidates. When faced with the comparison of just two candidates A and B, from the first assumption about preference rankings, voters will select the candidate whom they have ranked higher. So to get the vote total for candidate A, we simply count the number of ballots where candidate A is ranked above candidate B. Similarly we obtain the vote total for B.

(*a*) For the head-to-head comparison between Clinton and Bush we see that in columns 1, 2, and 5 Clinton is ranked above Bush for a total of $36+8+7 = 51$ votes. In columns 3, 4, and 6 Bush is ranked above Clinton for a total of $30+9+13 = 52$ votes. So Bush wins this head-to-head comparison 52 to 51. (This is the same result as the plurality with runoff).

(*b*) For the head-to-head comparison between Clinton and Perot we see that in columns 1, 2, and 4 Clinton is ranked above Perot for a total of $36+8+9 = 53$ votes. In columns 3, 5, and 6 Perot is ranked above Clinton for a total of $30+7+13 = 50$ votes. Clinton beats Perot.

(*c*) For the winner in a head-to-head comparison between Bush and Perot we see that in columns 2, 3, and 4 Bush is ranked above Perot for a total of $8+30+9 = 47$ votes. In columns 1, 5, and 6 Perot is ranked above Bush for a total of $36+7+13 = 56$ votes. Perot beats Bush.



This is an odd result: Voters prefer Bush to Clinton and Clinton to Perot, but voters do not prefer Bush to Perot. (This is an example where preferences are not *transitive*; see page 857 of your text.)

**DEFINITION 1.4** (Condorcet Method). A candidate who is the winner of a head-to-head comparison with every other candidate is called a **Condorcet winner**. A candidate who is the loser of a head-to-head comparison with every other candidate is called a **Condorcet loser**. A given election may or may not have a Condorcet winner and/or loser.

An important point is that there need not be a Condorcet winner or loser in a given election as we see in Example 1.7 above.

**EXAMPLE 1.8.** Determine whether there is a Condorcet winner and/or loser in the following election. Which candidate is the Borda winner?

| | 3 Voters | 2 Voters | 4 Voters |
|---|---|---|---|
| Alysha | 1 | 2 | 3 |
| Betsie | 2 | 1 | 2 |
| Cate | 3 | 3 | 1 |

**SOLUTION.** For the winner in a head-to-head comparison between Alysha and Betsie we see that in column 1 Alysha is ranked above Betsie for a total of 3 votes. In columns 2 and 3 Betsie is ranked above Alysha for a total of $2+4 = 6$ votes. Betsie beats Alysha.

For Alysha versus Cate we find Alysha gets $2+3 = 5$ votes while Cate gets just 4 votes, so Alysha beats Cate.

For Betsie versus Cate we find Betsie gets $3+2 = 5$ votes while Cate gets 4 votes, so Betsie beats Cate.

Thus, Betsie is a Condorcet winner since she beats both Alysha and Cate. Likewise Cate is a Condorcet loser since both other women beat her. Alysha is neither a Condorcet winner nor loser.

The Borda counts for the candidates are

$$\begin{aligned} \text{Alysha:} &\quad 3\times 1 + 2\times 2 + 4\times 3 = 19 \\ \text{Betsie:} &\quad 2\times 1 + 7\times 2 + 0\times 3 = 16 \\ \text{Cate:} &\quad 4\times 1 + 0\times 2 + 5\times 3 = 19. \end{aligned}$$

Betsie is also the Borda winner in this election. Note that Cate with 4 first-place votes is the plurality winner but she loses to Alysha in a plurality with runoff election. So all three candidates win under one type of election method.

☞ **Mindscape 1.8.** Return to Example 1.4 and Table 1.3. Is there a Condorcet winner? A Condorcet loser? Comment on the new results and the fact that Jones was the plurality winner and Smith the plurality with runoff winner.

## 1.8 Method of Pairwise Comparisons: Copeland's Method

We have seen that some elections, such as the 1992 presidential election as presented above, do not produce a Condorcet winner, i.e, there is no candidate that beats all others in head-to-head matchups. This possibility becomes more likely as the number of candidates in the election increases. However, there may be a candidate who wins more of the matchups than any other. This leads us to the method of comparisons (Copeland's method).

**DEFINITION 1.5** (Method of Pairwise Comparisons or Copeland's Method)**.** In each head-to-head comparison, the winner is assigned 1 point, the loser 0 points, and if there is a tie each candidate is assigned $\frac{1}{2}$ point. The overall winner of the election is the candidate with the most points after all head-to-head comparisons have taken place.

Obviously if there is a Condorcet winner in an election, then that person will also win by the method of pairwise comparisons. In Example 1.8, the Condorcet winner was Betsie. She would earn 2 points in the method of pairwise comparisons since she beats both Alysha and Cate. Alysha beats Cate (and loses to Betsie) so she has 1 point. Cate has no points since she loses to both opponents. Thus, Betsie with 2 points is the pairwise winner.

**EXAMPLE 1.9** (From Tannenbaum: *Excursions in Mathematics*)**.** Geneva has been given a franchise (the Generals) in the NFL. It is getting ready to make the number one pick in the upcoming player draft. The 22 coaches, scouts, and team executives have narrowed the list to five candidates: Allen, Byers, Castillo, Dixon, and Evans. Here are their preference ballots.

**TABLE 1.4.** Draft choice selection for the Geneva Generals

| | 2 Voters | 6 Voters | 4 Voters | 1 Voter | 1 Voter | 4 Voters | 4 Voters |
|---|---|---|---|---|---|---|---|
| Allen | 1 | 2 | 2 | 3 | 3 | 2 | 5 |
| Byers | 4 | 1 | 1 | 2 | 4 | 5 | 4 |
| Castillo | 3 | 3 | 5 | 1 | 1 | 4 | 2 |
| Dixon | 2 | 4 | 3 | 4 | 2 | 1 | 3 |
| Evans | 5 | 5 | 4 | 5 | 5 | 3 | 1 |

There are 10 possible pairwise comparisons to check. You should confirm that the following totals are correct.

| Comparison | Votes | Outcome |
|---|---|---|
| Allen versus Byers: | 7 votes to 15 votes | Byers gets 1 point |
| Allen versus Castillo: | 16 votes to 6 votes | Allen gets 1 point |
| Allen versus Dixon: | 13 votes to 9 votes | Allen gets 1 point |
| Allen versus Evans: | 18 votes to 4 votes | Allen gets 1 point |
| Byers versus Castillo: | 10 votes to 12 votes | Byers gets 1 point |
| Byers versus Dixon: | 11 votes to 11 votes | Byers gets $\frac{1}{2}$ point, Dixon gets $\frac{1}{2}$ point |
| Byers versus Evans: | 14 votes to 8 votes | Byers gets 1 point |
| Castillo versus Dixon: | 12 votes to 10 votes | Castillo gets 1 point |
| Castillo versus Evans: | 10 votes to 12 votes | Evans gets 1 point |
| Dixon versus Evans: | 18 votes to 4 votes | Dixon gets 1 point |

The final tally is Allen 3, Byers 2.5, Castillo 2, Dixon 1.5, and Evans 1. Notice that there is no Condorcet winner. However, by the method of pairwise comparisons, Allen is the winner and will be picked first by the Generals.

Example 1.9 shows that at least some elections which do not have Condorcet winners will still have winners by using the method of pairwise comparisons. Still there are some elections in which there is neither a Condorcet winner nor a winner by the method of pairwise comparisons. Check that Clinton, Bush, Perot election in Example 1.5 is an example of this situation.



## 1.9 Approval Voting

There also exist non-preferential voting methods. The one that we shall examine is approval voting. This method was first examined in detail in the 1970's. The premise of approval voting is that a candidate who most voters find 'acceptable' should win the election.

**DEFINITION 1.6** (Approval Voting)**.** With the **approval voting method**, voters indicate their approval or disapproval of each of the candidates. A ballot in an approval vote lists the candidates (or choices) and voters check off all the candidates of whom they approve. The winner is the candidate with the highest approval count.

Rather than ranking candidates, each voter must simply decide whether each particular candidate is acceptable. A voter's ballot consists of a series of checkmarks next to the names of all the candidates of whom they approve. Each voter may check as many or few candidates as they wish. The vote-for-two method described in Example 1.2 is a simple example of this method. Approval voting is now used to elect the Secretary General of the United Nations and the officers of the American Mathematical Society, as well as many other academic and professional societies.

**EXAMPLE 1.10.** Five professors are running for chair of the Mathematics and Computer Science Department: Belding, Critchlow, Eck, Oaks, and Vaughn. There are 11 members of the Department and the Department uses approval voting. The results are listed in Table 1.5. Which candidate is

**TABLE 1.5.** Chair election for Math/CS

| | 1 Voter | 1 Voter | 3 Voters | 1 Voter | 2 Voters | 1 Voter | 2 Voters |
|---|---|---|---|---|---|---|---|
| Belding | | | ✓ | | | ✓ | ✓ |
| Critchlow | ✓ | | ✓ | ✓ | | ✓ | ✓ |
| Eck | | | | | ✓ | | ✓ |
| Oaks | | | | ✓ | | | |
| Vaughn | | ✓ | | | ✓ | ✓ | ✓ |

the winner of the election?

**SOLUTION.** The approval vote counts are: Belding $3+1+2=6$, Critchlow 8, Eck 4, Oaks 1, and Vaughn 6. Critchlow is the winner.

**EXAMPLE 1.11.** Return to Example 1.8. Suppose that the voters not only ranked Alysha, Betsie, and Cate but also indicated their approval of the candidates, as shown in the following table. Who is the winner of the election using approval methods?

| | 3 Voters | 2 Voters | 4 Voters |
|---|---|---|---|
| Alysha | 1 ✓ | 2 ✓ | 3 |
| Betsie | 2 | 1 ✓ | 2 |
| Cate | 3 | 3 | 1 ✓ |

**SOLUTION.** It is easy to see that Alysha wins with 5 votes, while Betsie has 2 and Cate 4. Recall that Betsie was the Condorcet and Borda winner and Cate the plurality winner.

☞ **Mindscape 1.9** (From Gilbert and Hatcher: *Mathematics Beyond the Numbers*). The members of a community theater organization must vote to decide which play they would like to put on. The preference rankings and approvals of the members are given below.

| | 2 Voters | 1 Voter | 4 Voters | 1 Voter | 1 Voter | 3 Voters | 1 Voter | 2 Voters |
|---|---|---|---|---|---|---|---|---|
| *The Fantasticks* | 1 ✓ | 1 ✓ | 2 ✓ | 3 | 4 | 2 ✓ | 3 | 4 |
| *Romeo & Juliet* | 3 ✓ | 4 | 1 ✓ | 1 ✓ | 1 ✓ | 4 | 2 ✓ | 3 |
| *Our Town* | 4 | 3 | 3 | 4 | 2 ✓ | 1 ✓ | 1 ✓ | 2 ✓ |
| *Death of a Salesman* | 2 ✓ | 2 ✓ | 4 | 2 | 3 ✓ | 3 | 4 | 1 ✓ |

(*a*) What play is the plurality winner?

(*b*) What play is the plurality with instant runoff (between the top two finishers) winner?

(*c*) What play would win using the Borda method?

(*d*) What play is the Condorcet winner, if any? A Condorcet loser?

(*e*) What play is the approval method winner?

## 1.10 Fairness: What Do We Mean By Fair?

Now that we have examined several ways to hold an election we might ask, "What's the fairest way to hold an election?" The plurality method, which we often use in this country, when there are more two candidates may produce a winner that is viewed as undesirable by the majority of voters. A Condorcet winner who beats every other candidate head-to-head might seem a natural choice because voters prefer this candidate to every other. But we have seen Condorcet winners that do not even have a plurality of the votes, let alone a majority. And we have seen elections which do not even have a Condorcet winner.

☞ **Mindscape 1.10.** What are some of the other problems that arise with other voting methods?

So is one of these methods best or is there some other better method? To decide, we now list a set of fairness criteria for preference ballot elections. Some of these criteria are the bases for the various voting methods we have discussed. Others seem very reasonable, though you might not think of them at first. These criteria are not set in stone; political scientists and mathematicians continue to debate whether these or other criteria should be used.

For an election method to be fair, all of the following criteria should hold.

**Majority Criterion.** *A candidate receiving a majority of first-place votes should be the winner.* This seems self-evident. It would be unfair for a candidate with a majority of first-place votes to not win. (This is the **Go Along with the Consensus** criteria on page 860 of your text.)

**The Condorcet Criterion.** *A candidate who wins head-to-head matchups with all other candidates should be the winner.* When one candidate is preferred to all the others in head-to-head comparisons, how could it be fair for another candidate to win?



**The Monotonicity Criterion.** *Suppose an election is held and X is the winner. However, for some reason or another, there is a new election. If the only changes in the ballots are changes in favor of the winning candidate X (and only in favor of X), then X should remain the winner in the revote.* If X gets more votes or is ranked higher in the revote, how can it be fair for X to lose now? This is essentially the **Better is Better Principle** on page 855 of your text.

**The Independence of Irrelevant Alternatives Criterion (IIA).** *Suppose an election is held and X is the winner. However, for some reason, there is a new election. If the only changes are that one of the other losing candidates withdraws or is disqualified, then X should remain the winner in the revote.* It would be unfair to penalize the winner of an election because of the mere fact that one of the losers scratches out of the contest. (This is the **Ignore the Irrelevant** criteria on page 860 of you text.)

☞ **Mindscape 1.11.** The Independence of Irrelevant Alternatives (IIA) Criterion is often the most controversial of the 'fairness' criteria listed. Explain why that might be the case.

☞ **Mindscape 1.12.** Search on line to find at least one other 'fairness criteria' not listed above.

While all these criteria may seem reasonable, we now show that each of the preferential voting methods that we have discussed fails to satisfy at least one criterion.

**EXAMPLE 1.12** (Plurality and Plurality with Runoff Both Violate the Condorcet Criterion). Return to Example 1.8 which had the following election results.

| | 3 Voters | 2 Voters | 4 Voters |
|---|---|---|---|
| Alysha | 1 | 2 | 3 |
| Betsie | 2 | 1 | 2 |
| Cate | 3 | 3 | 1 |

The plurality winner is Cate with 4 votes. However, Cate loses to Alysha in a plurality with runoff election. But we saw that (check this again) Betsie is actually the Condorcet winner since she beats both Alysha and Cate in head-to-head matchups. So both plurality and plurality with runoff violate the Condorcet criterion.

**EXAMPLE 1.13** (Borda Count Violates the Majority Criterion.). All that we need to do is 'construct' an election where one candidate gets a slim majority, but another candidate gets most of the remaining first-place votes and lots of second-place votes.

| | 5 Voters | 2 Voters | 1 Voter | 1 Voter |
|---|---|---|---|---|
| Arturo | 1 | 3 | 3 | 2 |
| Dwayne | 2 | 1 | 2 | 1 |
| Carl | 3 | 2 | 1 | 3 |

Arturo is the winner of the majority of the votes (5 of 9). However, the the Borda counts for the candidates (check this) are: Arturo 16, Dwayne 15, and Carl 23. So Dwayne is the Borda winner which violates the majority criterion.

**EXAMPLE 1.14** (Plurality Violates IIA.)**.** Consider the following election results where the winner is to be determined by the plurality method.

| | 2 Voters | 2 Voters | 3 Voters |
|---|---|---|---|
| Jones | 2 | 1 | 2 |
| Johns | 3 | 2 | 1 |
| Smith | 1 | 3 | 3 |

Johns is the plurality winner with 3 first-place votes; each of the other candidates has 2. Tragically Smith (one of the losers) dies on election day before the votes are officially tallied. Using the preference rankings in the table above, the final vote becomes

| | 2 Voters | 2 Voters | 3 Voters |
|---|---|---|---|
| Jones | 1 | 1 | 2 |
| Johns | 2 | 2 | 1 |
| Smith | x | x | x |

Now the plurality winner is Jones 4 to 3 over Johns. Even though the relative preferences did not change (anyone preferring Johns initially still does in the 'revote'), using the plurality method the outcome of the election changed.

The Condorcet method certainly satisfies the Condorcet Criterion and it is not too hard to show that it satisfies the Majority Criterion. The problem with the Condorcet method is that in many elections there is no Condorcet winner. So it cannot be used as a method. We also considered a modification of the Condorcet method, namely the method of pairwise comparisons. However, we now show that it also violates at least one of the fairness criteria.

**EXAMPLE 1.15** (The Method of Pairwise Comparisons Violates IIA)**.** Return to Example 1.9 where we saw that using the method of pairwise comparisons, the Geneva Generals staff decided to pick Allen first in the NFL draft. However, it turns out that before the actual draft could take place, Castillo decided to take a full scholarship to graduate school in mathematics. Since he was not the top choice, we would not expect this to change the fact that Allen was the number one pick. Here's the revised table of preferences (compare it to Table 1.4).

**TABLE 1.6.** Draft choice selection for the Geneva Generals without Castillo

| | 2 Voters | 6 Voters | 4 Voters | 1 Voter | 1 Voter | 4 Voters | 4 Voters |
|---|---|---|---|---|---|---|---|
| Allen | 1 | 2 | 2 | ~~3~~ 2 | ~~3~~ 2 | 2 | ~~5~~ 4 |
| Byers | ~~4~~ 3 | 1 | 1 | ~~2~~ 1 | ~~4~~ 3 | ~~5~~ 4 | ~~4~~ 3 |
| ~~Castillo~~ | ~~3~~ | ~~3~~ | ~~5~~ | ~~1~~ | ~~1~~ | ~~4~~ | ~~2~~ |
| Dixon | 2 | ~~4~~ 3 | 3 | ~~4~~ 3 | ~~2~~ 1 | 1 | ~~3~~ 2 |
| Evans | ~~5~~ 4 | ~~5~~ 4 | 4 | ~~5~~ 4 | ~~5~~ 4 | 3 | 1 |

Now there are only 6 possible pairwise comparisons to check (there were 10 originally). You should confirm that the following totals are correct. In fact, check that the vote totals are *exactly the same* as in the earlier comparison with Castillo still in the mix. The reason is simple, removing Castillo does not change whether one of the candidates is ranked above or below the other. Their relative positions remain the same. Nonetheless, the final tally will change because there are fewer total comparisons—every candidate was originally compared to Castillo.



| Comparison | Votes | Outcome |
|---|---|---|
| Allen versus Byers: | 7 votes to 15 votes | Byers gets 1 point |
| Allen versus Dixon: | 13 votes to 9 votes | Allen gets 1 point |
| Allen versus Evans: | 18 votes to 4 votes | Allen gets 1 point |
| Byers versus Dixon: | 11 votes to 11 votes | Byers gets $\frac{1}{2}$ point, Dixon gets $\frac{1}{2}$ point |
| Byers versus Evans: | 14 votes to 8 votes | Byers gets 1 point |
| Dixon versus Evans: | 18 votes to 4 votes | Dixon gets 1 point |

The final tally is Allen 2, Byers 2.5, Dixon 1.5, and Evans 0. By the method of pairwise comparisons, Byers is the winner and will be picked first by the Generals. The results have changed because a loser dropped out, violating the IIA criterion.

☞ **Mindscape 1.13.** Using the vote-for-two method create a table of election results with candidates Goldman, Silverman, and Snowman which violates the majority criterion. Your table will need to list both a preferred candidate and an acceptable second candidate for each voter.

So far we have seen that plurality, plurality with instant runoff, Borda count, Condorcet method, and pairwise comparison all violate some aspect of fairness (or do not produce a winner). Is there some method that does satisfy all the fairness criteria?

The Nobel prize winning mathematical economist Kenneth Arrow began to exam such questions in the 1940's. In 1949 he completed his Ph.D. thesis at the University of Chicago entitled *Social Choice and Individual Values*. Using mathematical game theory he proved what is now called Arrow's Impossibility Theorem. This result says that *there is no voting method based on rankings that satisfies the four fairness criteria*. Though the criteria Arrow used were slightly different than the ones we have discussed, the same is true for our four criteria. In other words: *Perfect democratic voting is, not just in practice but in theory, impossible.* Read more about this result in your text, pages 858–863.

## 1.11 Homework Mindscapes

**1.** Look up the following information. Carefully and appropriately note your sources.

(*a*) How many votes did Bush get in Florida in the 2000 presidential election? Gore? Nader?

(*b*) In this race, Nader was the most liberal candidate, then Gore, while Bush was the most conservative. So it is likely that if Nader had not been on the ballot, the majority of his supporters would have voted for Gore. Assume that Nader did not run and that just a very small majority of his votes went to Gore. In particular, assume 50.5% of the votes Nader received went to Gore and 49.5% went to Bush. Who would have won Florida's electoral votes? [Most believe that Gore would have won substantially more of Nader's votes than this.]

(*c*) How would the Electoral College votes and national election have changed if Florida's electoral votes had gone to Gore?

**2.** The Hare method is another type of plurality method with a different runoff process. This method eliminates one candidate at a time rather than just having an instant runoff between the top two candidates.

**DEFINITION 1.7** (The Hare Method)**.** Each voter ranks the candidates in order of preference. The single candidate with the least number of first-place votes is eliminated and re-ranking takes place as needed. This elimination process continues until a candidate has a majority of first-place votes.

The Hare method is used in Australia, among other places, where it is sometimes called "first past the post" to use horse-racing language.

(*a*) Who wins the following NFL draft selection using the Hare method? (Caution: The data are slightly different than in Example 1.9.)

| | 2 Voters | 6 Voters | 4 Voters | 1 Voter | 2 Voters | 4 Voters | 5 Voters |
|---|---|---|---|---|---|---|---|
| Allen | 1 | 2 | 2 | 3 | 3 | 2 | 5 |
| Byers | 4 | 1 | 1 | 2 | 4 | 5 | 4 |
| Castillo | 3 | 3 | 5 | 1 | 1 | 4 | 2 |
| Dixon | 2 | 4 | 3 | 4 | 2 | 1 | 3 |
| Evans | 5 | 5 | 4 | 5 | 5 | 3 | 1 |

(*b*) Who wins using the instant runoff method described earlier?

**3.** Can there be two Condorcet winners in an election? How about two Condorcet losers? Explain your answer carefully.

**4.** Suppose that we have the following approval information for the HWS Board of Trustees election in Example 1.4.

| | 5 Voters | 4 Voters | 3 Voters | 3 Voters |
|---|---|---|---|---|
| Smith | 1 ✓ | 2 ✓ | 3 | 2 |
| Smythe | 2 ✓ | 1 ✓ | 2 | 3 |
| Jones | 3 | 3 | 1 ✓ | 1 ✓ |

(*a*) Who is the Borda winner?

(*b*) Who is the approval winner?

(*c*) How do these results compare with the Condorcet results and the plurality with runoff results?

**5.** Consider the following preference ballot results with for an election with four choices.

| | 6 Voters | 6 Voters | 4 Voters | 3 Voters | 4 Voters | 4 Voters |
|---|---|---|---|---|---|---|
| A | 1 | 4 | 4 | 1 | 1 | 4 |
| B | 4 | 1 | 2 | 3 | 2 | 3 |
| C | 2 | 2 | 3 | 2 | 4 | 2 |
| D | 3 | 3 | 1 | 4 | 3 | 1 |

(*a*) Who is the plurality winner? (Does this person have a majority of the votes?)

(*b*) Who is the plurality with instant runoff winner?

(*c*) Who is the Borda count winner?

(*d*) Is there a Condorcet winner? Is there a Condorcet loser?

(*e*) Who is the Hare winner?

(*f*) Suppose that all voters only 'approve' of those candidates rank that they ranked either 1 or 2. Put checkmarks in the table above and determine the approval winner.

(*g*) What are the results of a pairwise election?



**6.** (*a*) Explain why in an approval election if a voter approves of all of the candidates, then this voter's ballot does not affect the election outcome.

(*b*) Is this also true if the voter approves of none of the candidates?

**7.** This problem explores the Independence of Irrelevant Alternatives criterion. An election is decided using the method of pairwise comparisons. For some reason, one of the losing candidates, say Z, drops out before the results are made official. Suppose that A and B are two other candidates in this election.

(*a*) Suppose that a particular voter ranked A above B in the initial voting. Where is A ranked relative to B after candidate Z drops out? Hint: There are three cases to consider: Z was originally ranked above A, Z was originally ranked between A and B, or Z was originally ranked below B. What happens in each?

(*b*) Now consider the head-to-head matchup between A and B. How do the votes for A and B change once Z is removed from the election? Hint: Use part (a).

(*c*) Now consider A's final point tally of head-to-head wins. If A beat Z originally, how does A's tally change? If A tied Z originally, how does A's tally change? If A lost to Z originally, how does A's tally change? What is the maximum change in A's tally that can occur after Z drops out? When is the change positive?

(*d*) Now assume that A won the election with $n$ total points using the method of pairwise comparisons with Z in the race, but lost the election to X once Z dropped out. It turns out that there is only one way this can happen. What was A's point total with Z not in the race? Explain. What was X's point total with Z in the race? Explain. What was X's point total with Z not in the race? Explain. Did A beat Z? Did X beat Z?

(*e*) Under the same circumstances as in part (d), show that either A or X must have tied at least one other candidate.

(*f*) Return to the situation in Example 1.15. Is this what happened there?

**8.** Of the four fairness criteria listed in this article, the Independence of Irrelevant Alternatives (IIA) Criterion is the most controversial.

(*a*) Reconsider the 1968 election mentioned in Section 1.1 at the start of this article to explain why this criterion might be controversial. Suppose that George Wallace had dropped out of the race or was disqualified in some way. Would Nixon still have won as mandated by IIA? Hint: Consider Wallace's previous political association.

(*b*) Similarly use Example 1.4 to illustrate why the IIA criterion can be controversial. Given the information in the problem, what would happen in this election if Smythe (a plurality loser to Jones) dropped out in a revote?

**9.** (*a*) The Condorcet method was named after Nicolas de Caritat, marquis de Condorcet. Look him up and find out more about him. Carefully cite your sources.

(*b*) What is Condorcet's paradox? Explain carefully, again citing sources you use.

(*c*) The Borda count is named after Jean-Charles de Borda. Look him up and find out more about him. Carefully cite your sources.

(*d*) What was the relationship between Condorcet and Borda? Explain and cite your sources.

(*e*) What was the relationship between Borda and Napoleon Bonaparte? Explain and cite your sources.

**10.** Writing projects:

(*a*) What is the Gibbard-Satterthwaite Theorem?

(*b*) Read about tactical voting (also called strategic voting or sophisticated voting) and write a report.

(*c*) What is the modified Borda count election? Explain carefully and cite your sources.