Math 110: Assignment 14. Name ______________________________

Due Friday, April 30 at 2:00 pm. Special End of Term Time! Neatness counts. Use the back if needed.

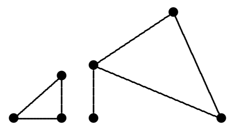
1. a) Making a point. Draw a connected graph G of your own choice below. Then next to it, draw it again and add a vertex in the middle of one edge, making two edges out of one. What happens to $\chi(G)$? Explain why?

b) Separation anxiety. Suppose that we have a connected graph that does NOT separate the plane into two or more regions (faces). Prove that the number of vertices is one more than the number of edges. [Hint: Use the Euler characteristic. What is f in this situation?]

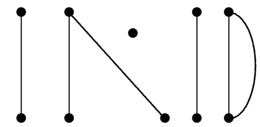
c) Dropped Connection. Can a connected graph G in the plane have 10 vertices and 8 edges? Explain carefully. (Hint: Use $\chi(G)$ and think about what f is here.)

d) An odd graph. Is it possible to draw a connected graph in the plane that has an odd number of vertices, but an even number of edges and an even number of faces? If so draw one; if not explain why.

2. Suppose that we now allow so-called **disconnected graphs**, i.e., graphs that are not connected. [Edges still should not cross each other, except where there is a vertex.] See the two examples below.



A 2 component graph.



A 5 component graph.

A graph has k **components** if it is composed of k separate (disjoint) connected subgraphs. For example, the graphs above have 2 and 5 components, respectively. A connected graph (like those we examined earlier) has just 1 component. Use the steps below to prove the following:

Theorem D. *If G is a graph in the plane with 2 components with no edges that cross, then* $\chi(G) = 3$.

a) Draw your own example of a graph with 2 components that has at least 10 vertices.

b) *Imagine* that G is *any* two component graph (not just the one you drew). Next *imagine* making a new graph H by adding a single edge that goes from a vertex of one component of G to a vertex of the other component of G. Draw such an edge in your example in a different color or with dashes.

c) How many components does the new graph H have? Explain.

d) What theorem tells us what the Euler characteristic of the new graph H is and what is this value?

e) How did the number of vertices change in going from G to H?

f) How did the number of edges change in going from G to H?

g) How did the number of faces change in going from G to H?

h) So dow did the Euler characteristic change in going from G to H? (Justify.)

i) You know the Euler characteristic of H. So what was the Euler characteristic of the original 2 component graph G?

j) Extra credit: Draw a graph G so that $\chi(G) = 5$. (Show your calculation.)

3. T-time. The subway system in Boston is called 'The T.' Go to http://www.mbta.com/schedules_and_maps/subway/ which shows the subway map for Boston: the Red, Orange, Green, Blue, and Silver Lines. (If you go to the online version of this assignment, you can just click on this link.) The thin purple lines are not part of the subway system. Without counting, **what is the Euler characteristic $\chi(T)$ for this map?** Briefly explain your reasoning.

$\chi(T) =$ ______________

4. Prove Theorem A. *If G is a connected graph in the plane with no edges that cross, then $\chi(G) = 2$.* Your work in class should be especially helpful here.

Proof: Suppose that G is a connected graph. Let's construct G step-by-step. We begin by drawing one of the vertices. If we compute the Euler characteristic right now at the start of the construction, then

$$\chi = v - e + f = \_\_\_\_ - \_\_\_\_ + \_\_\_\_ = \_\_\_\_.$$

At the next step we have to add an edge and its vertex so now

$$\chi = v - e + f = \_\_\_\_ - \_\_\_\_ + \_\_\_\_ = \_\_\_\_.$$

Notice that the Euler characteristic did not change and is still $\chi =$ ____. At each subsequent step when an edge is added, either (1) it connects two existing vertices or (2) it connects an existing vertex to a ________ vertex.

In case (1) here's how the vertices, edges, and faces change when the edge is added:

- the number of vertices ______________________________
- the number of edges ______________________________
- the number of faces ______________________________
- So the Euler characteristic stays the same because the new ____________and ____________cancel each other out.

In case (2) here's how the vertices, edges, and faces change when the edge is added:

- the number of vertices ______________________________
- the number of edges ______________________________
- the number of faces ______________________________
- So the Euler characteristic ______________________________

  because ______________________________.

Notice that in either case (1) or (2) the Euler characteristic ___________ the ___________ when an edge is added. Since the Euler characteristic __________________________ when new edges are added, then the Euler characteristic must be the same as when we started with the first vertex. Since the Euler characteristic starts out as ____, then it must still be ____ when the graph is complete. This means that: *If G is a connected graph in the plane, then $\chi(G) =$*____.