Infinite Limits

1-Minute Review: Large Numbers

It will be helpful to remember a couple of simple things about fractions or ratios. Let's use 0^+ to indicate a small positive number and 0^- to indicate a small-magnitude negative number. Then

- $\frac{\text{positive number}}{0^+}$ = large positive number
- $\frac{\text{positive number}}{0^-}$ = large-magnitude negative number
- $\frac{\text{negative number}}{0^+}$ = large-magnitude negative number
- $\frac{\text{negative number}}{0^-}$ = large positive number

For example, $\dfrac{1.2}{0^-}$ represents a large-magnitude negative number, while $\dfrac{-0.2}{0^-}$ represents a large positive negative number. Now let's look at an actual function.

EXAMPLE 7.1. Let $f(x) = \dfrac{1}{x-1}$. Examine the behavior of f near $x = 1$.

SOLUTION. $f(x)$ is a rational function the point $x = 1$ is not in its domain. However *near* $x = 1$, we can make a table of values and plot its graph.

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.9	-10	1.1	10
0.99	-100	1.01	100
0.999	-1000	1.001	1000
0.9999	-10000	1.0001	10000

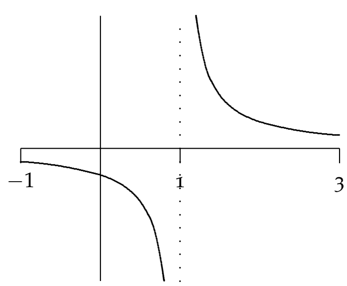


Figure 7.1: As $x \to 1^-$ we see $f(x) \to \frac{1}{0^-}$ and as $x \to 1^+$ we see $f(x) \to \frac{1}{0^+}$.

You may recognize the graph of $f(x)$ in Figure 7.1 as having a vertical asymptote at $x = 1$. We will have more to say about that soon. However, when we are interested in the values of a function *near* some point, we should realize that we are talking about limits. In the case of $f(x) = \frac{1}{x-1}$, clearly $\lim\limits_{x\to 1} \dfrac{1}{x-1}$ does not exist. However, we can still say *something useful* about the behavior of the function near 1. Because the values of $f(x)$ grow *arbitrarily large* or *increase without bound* as $x \to 1^+$, we write $\lim_{x\to 1^+} = \infty$. The infinity symbol means that $f(x)$ is getting large. It is not a number and the limit still does not exist in the original sense of the term. Likewise we write $\lim_{x\to 1^-} = -\infty$ because the values of $f(x)$ *decrease without bound* as $x \to 1^-$.

Infinite Limits

The following definitions of various infinite limits are informal, but for most situations in this course, they will be adequate.

DEFINITION 7.1 (Infinite Limits 'Informal'). Suppose that f is defined for all x near a. We write $\lim_{x\to a} f(x) = \infty$ and say that **the limit of** $f(x)$ **as** x **approaches** a **is infinity** if $f(x)$ becomes arbitrarily large for all x sufficiently close to (but not equal to) a.

We write $\lim_{x\to a} f(x) = -\infty$ and say that **the limit of** $f(x)$ **as** x **approaches** a **is negative infinity** if $f(x)$ is *negative* and becomes arbitrarily large in magnitude for all x sufficiently close to (but not equal to) a.

EXAMPLE 7.2. Determine $\lim_{x\to 0} \frac{2}{|x|}$.

SOLUTION. Notice that this limit does not exist in the usual sense since it is of the form $\frac{2}{0}$. Intuitively, we should recognize that the values of $\frac{2}{|x|}$ are becoming large in magnitude as $x \to 0$ We will deal with most infinite limits either graphically or informally. A table of values and a quick plot confirms this. In this case. Using Definition 7.1 we write.

$$\lim_{x\to 0} \frac{2}{|x|} = \infty.$$

$x < 1$	$f(x)$	$x > 1$	$f(x)$
-0.1	20	.1	20
-0.01	200	0.01	200
-0.001	2000	0.001	2000
-0.0001	20000	0.0001	20000

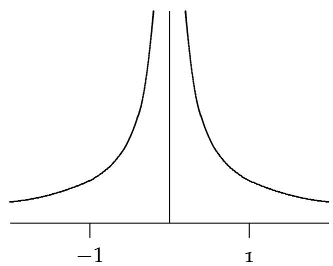


Figure 7.2: As $x \to 0$ we see $f(x)$ increases without bound so $\lim_{x\to 0} = \infty$.

In Example 7.2, f increased without bound from *both sides* of 0. In Example 7.1, f increased without bound from *right sides* of 1 and decreased without bound from *left side*. Naturally, this leads to the notion of one-sided infinite limits. There are four possible behaviors that can occur.

DEFINITION 7.2 (One-sided Infinite Limits). Suppose that f is defined for all x near a with $x > a$.

- We write $\lim_{x\to a^+} f(x) = \infty$ if $f(x)$ becomes arbitrarily large for all x sufficiently close to a with $x > a$.
- Similarly, we write $\lim_{x\to a^+} f(x) = -\infty$ if $f(x)$ is *negative* and becomes arbitrarily large in magnitude for all x sufficiently close to a with $x > a$.

Now suppose that f is defined for all x near a with $x < a$.

- We write $\lim_{x\to a^-} f(x) = \infty$ if $f(x)$ becomes arbitrarily large for all x sufficiently close to a with $x < a$.
- We write $\lim_{x\to a^-} f(x) = -\infty$ if $f(x)$ is *negative* and becomes arbitrarily large in magnitude for all x sufficiently close to a with $x < a$.

Warning: In Example 7.1 we saw that $\lim_{x\to 1^+} \frac{1}{x-1} = +\infty$ and $\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$. **These limits still do not exist in the usual sense.** $\pm\infty$ is just short-hand notation to describe the behavior of a function near a point.

Tip: Most infinite limits should be evaluated as one-sided limits. Often a function will exhibit very different behavior on either side of a point where the function is becoming unbounded.

Vertical Asymptotes

At a point where a function becomes large in magnitude, the graph appears almost vertical. To describe this, we use the following terminology.

DEFINITION 7.3 (Vertical Asymptote)**.** If $\lim_{x \to a^+} f(x) = \pm\infty$ or if $\lim_{x \to a^-} f(x) = \pm\infty$ we say that the line $x = a$ is a **vertical asymptote** of f.

Wherever there is an infinite limit, there is a vertical asymptote. The limit gives us a bit more information in that it specifies wether the function is *increasing* or *decreasing* with out bound. Knowing only that $x = a$ is a vertical asymptote only tells us that the function is unbounded. A familiar example will illustrate these ideas.

EXAMPLE 7.3. Let $f(x) = \tan x$ on the interval $[-2.5, 2.5]$ as shown below.

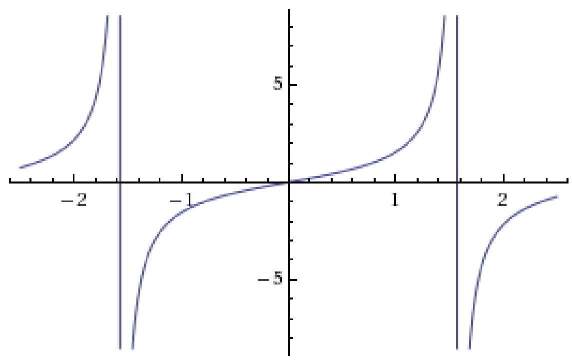


Figure 7.3: The graph of $\tan x$ and two of its vertical asymptotes.

Determine each of the following limits.

(a) $\lim_{x \to -\frac{\pi}{2}^-} f(x)$ $\quad (b)$ $\lim_{x \to -\frac{\pi}{2}^+} f(x)$ $\quad (c)$ $\lim_{x \to -\frac{\pi}{2}} f(x)$

(d) $\lim_{x \to 0^-} f(x)$ $\quad (e)$ $\lim_{x \to 0+} f(x)$ $\quad (f)$ $\lim_{x \to 0} f(x)$

(g) $\lim_{x \to \frac{\pi}{2}^-} f(x)$ $\quad (h)$ $\lim_{x \to \frac{\pi}{2}^+} f(x)$ $\quad (i)$ $\lim_{x \to \frac{\pi}{2}} f(x)$

(j) Where does $f(x)$ have vertical asymptotes?

Make sure you know the values of the trig functions at the standard angles, especially those in the first quadrant.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$-$

SOLUTION. We know that $\tan x = \dfrac{\sin x}{\cos x}$ is not defined at $\pm\frac{\pi}{2}$.

(a) Using the graph, the values of $\tan x$ are negative grow arbitrarily large and magnitude as x approaches $-\frac{\pi}{2}$ from the left. So $\lim_{x \to -\frac{\pi}{2}^-} \tan x = -\infty$. This already means that $x = -\frac{\pi}{2}$ is a vertical asymptote for $\tan x$.

(b) The values of $\tan x$ are grow arbitrarily large as x approaches $-\frac{\pi}{2}$ from the right. So $\lim_{x \to -\frac{\pi}{2}^+} \tan x = \infty$.

(c) Since the two one-sided limits at $-\frac{\pi}{2}$ differ, $\lim_{x \to -\frac{\pi}{2}} \tan x$ DNE.

(d) As x approaches 0 from either side, we see that $\tan x$ approaches 0 so $\lim_{x \to 0^-} \tan x = \lim_{x \to 0+} \tan x = \lim_{x \to 0} f(x) = 0$. Further $x = 0$ is not a vertical asymptote for $\tan x$.

(g) The values of $\tan x$ are negative grow arbitrarily large and magnitude as x approaches $\frac{\pi}{2}$ from the left. So $\lim_{x \to \frac{\pi}{2}^-} \tan x = -\infty$. This means that $x = \frac{\pi}{2}$ is a vertical asymptote for $\tan x$.

(h) The values of $\tan x$ are grow arbitrarily large as x approaches $\frac{\pi}{2}$ from the right. So $\lim_{x \to \frac{\pi}{2}^+} \tan x = \infty$.

(*i*) As with $-\frac{\pi}{2}$ we see $\lim_{x\to-\frac{\pi}{2}} \tan x$ DNE.

(*j*) There were two vertical asymptotes: at $x = \pm\frac{\pi}{2}$. More generally, $\tan x$ will have a vertical asymptote (VA) at any odd multiple of $\frac{\pi}{2}$

Determining Infinite Limits

Many of the infinite limits that we will need to determine will arise from rational functions or other functions that involve quotients. We look for infinite limits where such functions are not defined. This typically means locating points where the denominators of such functions are 0.

Since graphing may be time consuming (and since calculators sometimes give inaccurate graphs), we will use the simple analytic method outlined in Section 0 a few pages back. Let's try several examples.

EXAMPLE 7.4. Find the vertical asymptotes and infinite limits for $f(x) = \dfrac{x+1}{x-2}$.

SOLUTION. This is a rational function which is continuous for all x except where the denominator is 0, namely, $x = 2$. Examine the one-sided limits at 2.

$$\lim_{x\to 2^-} \frac{\overbrace{x+1}^{\to 3}}{\underbrace{x-2}_{\to 0^-}} = -\infty.$$

Analysis: The numerator approaches 3 while the denominator is *negative* and approaching 0. So the quotient is negative and large in magnitude. Similarly

$$\lim_{x\to 2^+} \frac{\overbrace{x+1}^{\to 3}}{\underbrace{x-2}_{\to 0^+}} = \infty.$$

Either one-sided limit shows that $x = 2$ is a VA.

EXAMPLE 7.5. Find the vertical asymptotes and infinite limits for $f(x) = \dfrac{x+3}{x^2(x-2)}$.

SOLUTION. This is a rational function which is continuous for all x except where the denominator is 0, which is at $x = 0$ and $x = 2$ So we examine the one-sided limits at $x = 1$ and $x = 2$.

Near $x = 2$:

$$\lim_{x\to 2^-} \frac{\overbrace{x+3}^{\to 5}}{\underbrace{x^2(x-2)}_{\to 4\cdot 0^+ = 0^+}} = \infty.$$

Similarly, the limit from the right is

$$\lim_{x\to 2^+} \frac{\overbrace{x+3}^{\to 5}}{\underbrace{x^2(x-2)}_{\to 4\cdot 0^- = 0^-}} = -\infty.$$

So $x = 2$ is a VA.

Near $x = 0$:

$$\lim_{x\to 0^-} \frac{\overbrace{x+3}^{\to 3}}{\underbrace{x^2(x-2)}_{\to 0^+(-2) = 0^-}} = -\infty.$$

Similarly, the limit from the right is

$$\lim_{x\to 0^+} \frac{\overbrace{x+3}^{\to 3}}{\underbrace{x^2(x-2)}_{\to 0^+(-2)-=0^-}} = -\infty.$$

So $x = 2$ is a VA.

EXAMPLE 7.6. Find the vertical asymptotes and infinite limits for $f(x) = \dfrac{x^2-4x+3}{x^2-3x+2}$.

SOLUTION. This is a rational function which is continuous for all x except where the denominator is 0, so we begin by factoring.

$$f(x) = \frac{x^2-4x+3}{x^2-3x+2} = \frac{(x-1)(x-3)}{(x-1)(x-2)}.$$

So we examine the one-sided limits at $x = 1$ and $x = 2$.

Near $x = 2$:

$$\lim_{x\to 2^-} \frac{(x-1)(x-3)}{(x-1)(x-2)} \overset{\text{Simplify}}{=} \lim_{x\to 2^-} \frac{\overbrace{x-3}^{\to -1}}{\underbrace{x-2}_{\to 0^-}} = \infty.$$

Similarly, the limit from the right is

$$\lim_{x\to 2^+} \frac{(x-1)(x-3)}{(x-1)(x-2)} \overset{\text{Simplify}}{=} \lim_{x\to 2^+} \frac{\overbrace{x-3}^{\to -1}}{\underbrace{x-2}_{\to 0^-}} = -\infty.$$

$x = 2$ is a VA.

f is also undefined at $x = 1$. However,

$$\lim_{x\to 1^-} \frac{(x-1)(x-3)}{(x-1)(x-2)} \overset{\text{Simplify}}{=} \lim_{x\to 1^-} \frac{\overbrace{x-3}^{\to -2}}{\underbrace{x-2}_{\to 1}} = -2.$$

In fact, both one-sided limits at $x = 1$ are equal and the two-sided limit is

$$\lim_{x\to 1} \frac{(x-1)(x-3)}{(x-1)(x-2)} \overset{\text{Simplify}}{=} \lim_{x\to 1} \frac{\overbrace{x-3}^{\to -2}}{\underbrace{x-2}_{\to 1}} = -2.$$

There is no VA at $x = 1$.

Take a look at the graph of $f(x)$ in Figure 7.4. Instead of a vertical asymptote at $x = 1$ there is a 'hole' in the graph. We give such points a special name.

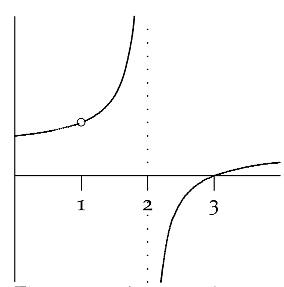


Figure 7.4: As $x \to 0$ we see $f(x)$ increases without bound so $\lim\limits_{x\to 0} = \infty$.

DEFINITION 7.4. (Removable Discontinuity) A function f has a **removable discontinuity** at a if the following hold:

1. $\lim\limits_{x\to a} f(x)$ exists (and is finite).

2. $\lim_{x\to a} f(x) \neq f(a)$. Note: $f(a)$ may not even exist.

Remember that f is continuous at a if $\lim_{x\to a} f(x) = f(a)$. So condition 2 in the definition ensures that f is NOT continuous at a. On the other hand, the function is well-behaved near a, since $\lim_{x\to a} f(x)$ exists. In fact, if we defined (or redefined) $f(a)$ to be $\lim_{x\to a} f(x)$, then f would be continuous. That is, we could *remove the discontinuity* by redefining f and filling in the hole in the graph (see Figure 7.4).

Again, returning to Figure 7.4, f is also discontinuous at $x = 2$. But since $\lim_{x\to 2}$ DNE, the discontinuity is not removable.

YOU TRY IT 7.1. Explain why if f has a VA at $x = a$, then the discontinuity at a is NOT removable.

Answer to YOU TRY IT 7.1 . To have a VA, either $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$. In either case, $\lim_{x\to a} f(x)$ does not exist in the standard sense (is not finite).

EXAMPLE 7.7. Determine the points at which $f(x) = \dfrac{x^2-5x+6}{x^3+x^2-12x}$ is discontinuous. At which points does f have VA's? Removable discontinuities?

SOLUTION. Since $f(x)$ is rational it is continuous at all points in its domain. So it will fail to be continuous where the denominator is equal to 0. So let's factor f:

$$f(x) = \frac{x^2-5x+6}{x^3+x^2-12x} = \frac{(x-2)(x-3)}{x(x+4)(x-3)}, \qquad x \neq -4, 0, 3.$$

f is discontinuous at -4, 0, and 3. Now examine appropriate limits to check for VA's and removable discontinuities. (Can you predict which are which?)

At $x = -4$:

$$\lim_{x\to -4^-} f(x) = \lim_{x\to -4^-} \frac{(x-2)(x-3)}{x(x+4)(x-3)} = \lim_{x\to -4^-} \frac{\overbrace{x-2}^{\to -6}}{\underbrace{x(x+4)}_{\to -4\cdot 0^- = 0^+}} = -\infty.$$

This is enough to conclude that f has a VA at -4. Caution: Take care with the calculation of the sign in the denominator.

At $x = 0$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{(x-2)(x-3)}{x(x+4)(x-3)} = \lim_{x\to 0^-} \frac{\overbrace{x-2}^{\to -2}}{\underbrace{x(x+4)}_{\to 0^-\cdot 4 = 0^-}} = -\infty.$$

This is enough to conclude that f has a VA at 0. Again: Take care with the calculation of the sign in the denominator.

At $x = 3$: Having seen the factorization of f, we know that we can calculate a two-sided limit at 3.

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{(x-2)(x-3)}{x(x+4)(x-3)} = \lim_{x\to 3} \frac{x-2}{x(x+4)} = \frac{1}{21}.$$

Since $\lim_{x\to 3} f(x)$ exists but $f(3)$ is not defined, then f has a removable discontinuity at $x = 3$.

YOU TRY IT 7.2. Determine $\lim_{x\to -4^+} f(x)$ and $\lim_{x\to 0^+} f(x)$ for the function in Example 7.7.

Answer to YOU TRY IT 7.2 : Both are ∞.

EXAMPLE 7.8. Determine the points at which $f(x) = \dfrac{\frac{1}{x-2} - \frac{1}{2}}{x-4}$ is discontinuous. At which points does f have VA's? Removable discontinuities?

SOLUTION. Since $f(x)$ is rational it is continuous at all points in its domain. We can see immediately that f is not defined at $x = 4$ and $x = 2$, where there would be division by 0, so f is not continuous at these two points. Let's simplify the expression for f before taking the appropriate limits.

$$f(x) = \frac{\frac{1}{x-2} - \frac{1}{2}}{x-4} = \frac{\frac{2-(x-2)}{2(x-2)}}{x-4} = \frac{4-x}{2(x-2)(x-4)}.$$

At $x = 2$:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{4-x}{2(x-2)(x-4)} = \lim_{x\to 2^-} \frac{-1}{2(x-2)} = \lim_{x\to 2^-} \frac{-1}{\underbrace{2(x-2)}_{\to 2\cdot 0^- = 0^-}} = \infty.$$

This is enough to conclude that f has a VA at 2.

At $x = 4$: Having seen the factorization of f, we know that we can calculate a two-sided limit at 4.

$$\lim_{x\to 4} f(x) = \lim_{x\to 4} \frac{4-x}{2(x-2)(x-4)} = \lim_{x\to 4} \frac{-1}{2(x-2)} = -\frac{1}{8}.$$

Since $\lim_{x\to 4} f(x)$ exists but $f(4)$ is not defined, then f has a removable discontinuity at $x = 4$.

YOU TRY IT 7.3. Determine $\lim_{x\to 2^+} f(x)$ for the function in Example 7.8.

Answer to YOU TRY IT 7.3 : $-\infty$.

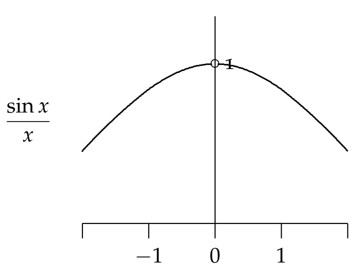
YOU TRY IT 7.4. Determine where the function $f(x) = \dfrac{x^2-1}{x^2-3x+2}$ has vertical asymptotes and where it has removable discontinuities.

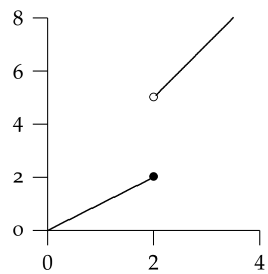
Answer to YOU TRY IT 7.4 : VA at $x = 2$. Removable discontinuity at $x = 1$ since $\lim_{x\to 1} f(x) = -2$ and $f(a)$ UND.

YOU TRY IT 7.5. Consider the two graphs below that we saw earlier this term when first considering limits. Discuss the type of discontinuity in each.

Answer to YOU TRY IT 7.5 : Since $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ but $\dfrac{\sin x}{x}$ is not defined at 0, then by Definition 7.4 there is a removable discontinuity at 0.

On the right $\lim_{x\to 2} f(x)$ DNE because the two one-sided derivatives are not equal. So the discontinuity at 2 is not removable.

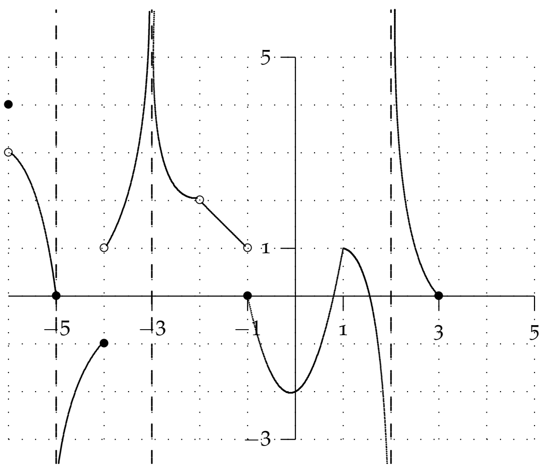




$$f(x) = \begin{cases} x, & \text{if } x \le 2 \\ 2x+1, & \text{if } x > 2 \end{cases}$$

7.0 Problems

1. Use the graph of f to evaluate each of the expressions in the table or explain why the value does not exist. For the "Cont(inuity)" column, your answer should be: , and "Yes" if it is continuous, "Rem" if it has a removable discontinuity, and "No" if it is not continuous and does not have a removable discontinuity. For the last few points, complete the graph so that the given information is true. (When the graph goes off the grid, the function is becoming unbounded.)



a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$	Cont	VA
-6						
-5						
-4						
-3						
-2						
-1						
0						
1						
2						
3					Yes	
4				5		Yes

2. (*a*) True or false: Even if $f(x)$ has a removable discontinuity at $x = a$ it still might have a VA at $x = a$. Explain.

(*b*) Give the equation of a rational function with a VA at $x = -2$ and a removable discontinuity at $x = 6$.

3. Here are five straightforward one-sided limits problems. Use $+\infty$ or $-\infty$ if appropriate. When the denominator goes to 0 but the numerator does not, determine the signs of each and then determine the limit. For indeterminate $\frac{0}{0}$ limits do more work.

(*a*) $\lim_{x\to 3^+} \dfrac{-2x+1}{x-3}$

(*b*) $\lim_{x\to -4^-} \dfrac{x^2+3x-4}{x+4}$

(*c*) $\lim_{x\to -2^+} \dfrac{x^2-4}{x+4}$

(*d*) $\lim_{x\to 1^+} \dfrac{x^2}{x^2-1}$

(*e*) $\lim_{x\to 2^-} \dfrac{x^2}{x(x-2)}$

4. Which of the functions above have a VA at the point in the limit? How can you tell?

5. This last problem is more like a test question.

(*a*) $f(x) = \dfrac{2x-8}{x^2-4x}$ is continuous except at two points: $x =$ ________, ________ because f is ______________________________.

(*b*) At each point from part (a), determine the limit. If infinite limits are required check both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$.

(*c*) Does f have a VA at either point? Explain.

(*d*) Does f have a Removable Discontinuity (RD) at either point? Explain.

6. Determine where each of these functions is continuous. Does either have any VA's? Removable discontinuities. Show your work.

(*a*) $g(x) = \dfrac{x^2-2x-8}{x^2-16}$ (*b*) $f(x) = \dfrac{\frac{1}{x-4}+\frac{1}{2}}{x-2}$