38.6 Extra Fun: The Indeterminate Forms 1^∞, ∞^0, and 0^0

Some of the most interesting limits in elementary calculus have the indeterminate forms 1^∞, 0^0, or 0^0. All of these indeterminate limit forms arise from functions that have both a variable base and a variable exponent (power). For example, consider

$$\begin{aligned} \lim_{x\to 0^+} x^x & \qquad \text{Form: } 0^0 \\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x & \qquad \text{Form: } 1^\infty \\ \lim_{x\to\infty} x^{1/x}) & \qquad \text{Form: } \infty^0 \end{aligned}$$

We will use logs and l'Hôpital's rule to simplify some of these limit calculations.

General Form The general form of all of these limits is $\lim_{x\to a}[f(x)]^{g(x)} = y$. To simplify these limits we use the natural log to *undo* the power. If the eventual limit is y (which is unknown to us—it's what we are trying to find, then

$$y = \lim_{x\to a}[f(x)]^{g(x)}.$$

We take the natural log of both sides—here $[f(x)]^{g(x)}$ is assumed to be positive.

$$\ln y = \ln(\lim_{x\to a}[f(x)]^{g(x)})$$

As long as $f(x)$ and $g(x)$ are continuous, we can switch the order of the log and the limit and use log properties

$$\begin{aligned} \ln y &= \lim_{x\to a} \ln([f(x)]^{g(x)}) \\ \ln y &= \lim_{x\to a} g(x)\ln(f(x)) \end{aligned}$$

At this stage we typically use l'Hôpital's rule to find the limit, call it L. Then $\ln y = L$ so we must have $y = e^L$. Let's look at some examples.

EXAMPLE 38.7. Determine $\lim_{x\to 0^+}(2x)^x$. Notice that this is a 0^0 form.

SOLUTION. Let $y = \lim_{x\to 0^+}(2x)^x$. We want to find y. Using the log process above,

$$\begin{aligned} \ln y &= \ln(\lim_{x\to 0^+}(2x)^x) \\ \ln y &= \lim_{x\to 0^+} \ln(2x)^x \\ \ln y &= \lim_{x\to 0^+} x\ln 2x \\ \ln y &= \lim_{x\to 0^+} \frac{\ln 2x}{\frac{1}{x}} \\ \ln y &\overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{\frac{2}{2x}}{-\frac{1}{x^2}} \\ \ln y &= \lim_{x\to 0^+} -x \\ \ln y &= 0. \end{aligned}$$

But $\ln y = 0$ implies $y = e^0 = 1$. So $\lim_{x\to 0^+}(2x)^x = y = 1$. Wow!

EXAMPLE 38.8 (Critical Example). Determine $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. Notice that this is a 1^∞ form.

SOLUTION. Let $y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. We want to find y. Using the log process,

$$\begin{aligned}
\ln y &= \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]\\
\ln y &= \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x\\
\ln y &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)\\
\ln y &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}\\
\ln y &= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}\\
\ln y &= 1.
\end{aligned}$$

But $\ln y = 1$ implies $y = e^1 = e$. So $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = y = e$. Double Wow!! In fact, in some courses you will see that e is defined this way.

EXAMPLE 38.9 (Critical Example). Determine $\lim_{x\to\infty} x^{1/x}$. Notice that this is a ∞^0 form.

SOLUTION. Let $y = \lim_{x\to\infty} x^{1/x}$. We want to find y. Using the log process,

$$\begin{aligned}
\ln y &= \ln\lim_{x\to\infty} x^{1/x}\\
\ln y &= \lim_{x\to\infty}\ln x^{1/x}\\
\ln y &= \lim_{x\to\infty}\frac{1}{x}\ln x\\
\ln y &= \lim_{x\to\infty}\frac{\ln x}{x}\\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1}\\
\ln y &= \frac{0}{1} = 0.
\end{aligned}$$

But $\ln y = 0$ implies $y = e^0 = 1$. So $\lim_{x\to\infty} x^{1/x} = y = 1$. Neat!

EXAMPLE 38.10. Determine $\lim_{x\to\infty}(e^x+x)^{2/x}$. Notice that this is a ∞^0 form again.

SOLUTION. As usual let $y = \lim_{x\to\infty}(e^x+x)^{2/x}$, so

$$\begin{aligned}
\ln y &= \ln\lim_{x\to\infty}(e^x+x)^{2/x}\\
\ln y &= \lim_{x\to\infty}\ln(e^x+x)^{2/x}\\
\ln y &= \lim_{x\to\infty}\frac{2}{x}\ln(e^x+x)\\
\ln y &= \lim_{x\to\infty}\frac{2\ln(e^x+x)}{x}\\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to\infty}2\frac{\frac{e^x+1}{e^x+x}}{1}\\
\ln y &= \lim_{x\to\infty}2\frac{e^x+1}{e^x+x}\\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to\infty}2\frac{e^x}{e^x+1}\\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to\infty}2\frac{e^x}{e^x}\\
\ln y &= 2.
\end{aligned}$$

But $\ln y = 2$ implies $y = e^2$. So $\lim_{x\to\infty} (e^x + x)^{2/x} = e^2$.

EXAMPLE 38.11. Determine $\lim_{x\to 2^+} [5(x-2)]^{x-2}$. Notice that this is a 0^0 form.

SOLUTION. Let $y = \lim_{x\to 2^+} [5(x-2)]^{x-2}$. We want to find y. Using the log process above,

$$\begin{aligned}
\ln y &= \ln\big(\lim_{x\to 2^+} [5(x-2)]^{x-2} \\
\ln y &= \lim_{x\to 2^+} \ln[5(x-2)]^{x-2} \\
\ln y &= \lim_{x\to 2^+} (x-2)\ln[5(x-2)] \\
\ln y &= \lim_{x\to 2^+} \frac{[5(x-2)]}{\frac{1}{x-2}} \\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to 2^+} \frac{\frac{5}{x-2}}{-\frac{1}{(x-2)^2}} \\
\ln y &= \lim_{x\to 2^+} -5(x-2) \\
\ln y &= 0.
\end{aligned}$$

But $\ln y = 0$ implies $y = e^0 = 1$. So $\lim_{x\to 2^+} [5(x-2)]^{x-2} = y = 1$. This is becoming routine.

YOU TRY IT 38.3. Here's a great problem to see if you have mastered these ideas. Determine Determine $\lim_{x\to 0^+} [\sin(x)]^x$.

Answer to YOU TRY IT 38.3 : 1. Hint: Use l'Hôpital's rule twice.

38.7 *Indeterminate Form* $\infty - \infty$

This situation arises when both $f(x)$ and $g(x)$ are functions going to infinity as $x \to a$. Since the functions can approach infinity at very different rates, we cannot say for sure what such a limit will be—and we certainly cannot conclude that the limit is 0! Some examples will illustrate this.

EXAMPLE 38.12. Determine $\lim_{x\to 1^+} \frac{4}{x^2-1} - \frac{2}{x-1}$. Notice both terms are going to $+\infty$ as $x \to 1^+$ since the denominators in each are going to 0^+.

SOLUTION. For this first one we begin by using a common denominator and then use l'Hôpital's rule. (Do you see why l'Hôpital's rule applies?)

$$\lim_{x\to 1^+} \frac{4}{x^2-1} - \frac{2}{x-1} = \lim_{x\to 1^+} \frac{4-2(x+1)}{x^2-1} \overset{\text{l'Ho}}{=} \lim_{x\to 1^+} \frac{-2}{2x} = \frac{-2}{2} = -1.$$

That was easy.

EXAMPLE 38.13. Determine $\lim_{x\to\infty} \ln(6x+1) - \ln(2x+7)$. Notice both terms are going to $+\infty$ as $x \to \infty$.

SOLUTION. For this first one we begin by using a common log property then switch the limit and the log (continuity) and then use l'Hôpital's rule. (Check that l'Hôpital's rule applies at the appropriate time.)

$$\begin{aligned}
\lim_{x\to\infty} \ln(6x+1) - \ln(2x+7) = \lim_{x\to\infty} \ln\left(\frac{6x+1}{2x+7}\right) = \ln\left(\lim_{x\to\infty} \frac{6x+1}{2x+7}\right) &\overset{\text{l'Ho}}{=} \ln \lim_{x\to\infty} \frac{6}{2} \\
&= \ln 3.
\end{aligned}$$

Not bad.

EXAMPLE 38.14. Determine $\lim_{x\to 1^+} \frac{4}{\ln x} - \frac{4}{x-1}$. Notice both terms are going to $+\infty$ as $x \to 1^+$ since the denominators in each are going to 0^+.

SOLUTION. For this first one we begin by using a common denominator and then use l'Hôpital's rule. (Check l'Hôpital's rule applies when used.)

$$\begin{aligned}
\lim_{x\to 1^+} \frac{4}{\ln x} - \frac{4}{x-1} = \lim_{x\to 1^+} \frac{4(x-1) - 4\ln x}{(x-1)\ln x} &\overset{\text{l'Ho}}{=} \lim_{x\to 1^+} \frac{4 - \frac{4}{x}}{\ln x + \frac{x-1}{x}} \\
&= \lim_{x\to 1^+} \frac{\frac{4x-4}{x}}{\frac{x\ln x + x - 1}{x}} \\
&= \lim_{x\to 1^+} \frac{4x-4}{x\ln x + x - 1} \\
&\overset{\text{l'Ho}}{=} \lim_{x\to 1^+} \frac{4}{\ln x + \frac{x}{x} + 1} \\
&= \frac{4}{0+1+1} \\
&= 2.
\end{aligned}$$

That was not so easy.

EXAMPLE 38.15. Determine $\lim_{x\to 3^+} \frac{1}{x^2-9} - \frac{\sqrt{x-2}}{x^2-9}$. Notice both terms are going to $+\infty$ as $x \to 3^+$ since the denominators in each are going to 0^+.

SOLUTION. The common denominator this time is obvious. Eventually use l'Hôpital's rule. (Check l'Hôpital's rule applies when used.)

$$\lim_{x\to 3^+} \frac{1}{x^2-9} - \frac{\sqrt{x-2}}{x^2-9} = \lim_{x\to 3^+} \frac{1-\sqrt{x-2}}{x^2-9} \overset{\text{l'Ho}}{=} \lim_{x\to 3^+} \frac{-\frac{1}{2\sqrt{x-2}}}{2x} = \frac{-\frac{1}{2}}{6} = -\frac{1}{12}.$$

EXAMPLE 38.16. Determine $\lim_{x\to\infty} 2\ln(x+1) - \ln(2x^2+7)$. Notice both terms are going to $+\infty$ as $x \to \infty$.

SOLUTION. For this first one we begin by using two log properties then switch the limit and the log (continuity) and then use l'Hôpital's rule. (Check that l'Hôpital's rule applies at the appropriate time.)

$$\begin{aligned}
\lim_{x\to\infty} 2\ln(x+1) - \ln(2x^2+7) &= \lim_{x\to\infty} \ln(x+1)^2 - \ln(2x^2+7) \\
&= \lim_{x\to\infty} \ln(x^2+2x+1) - \ln(2x^2+7) \\
&= \lim_{x\to\infty} \ln\left(\frac{x^2+2x+1}{2x^2+7}\right) \\
&= \ln\left(\lim_{x\to\infty} \frac{x^2+2x+1}{2x^2+7}\right) \\
&\overset{\text{l'Ho}}{=} \ln \lim_{x\to\infty} \frac{2x+2}{4x} \\
&\overset{\text{l'Ho}}{=} \ln \lim_{x\to\infty} \frac{2}{4} \\
&= \ln\frac{1}{2}.
\end{aligned}$$

38.8 Problems

1. Some interesting limits. Answers not in order: 0, 0, 0, $\frac{1}{2}$, $\ln 4$, 1, 1, 2, e^2, e^k, $+\infty$, $-\infty$, and -6,

(*a*) $\displaystyle\lim_{x\to 0} \frac{\cos 4x - \cos 2x}{x^2}$ (*b*) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$ (*c*) $\displaystyle\lim_{x\to 0^+} 2x \ln x$

(*d*) $\displaystyle\lim_{x\to\infty} x^2 e^{-x}$ (*e*) $\displaystyle\lim_{x\to 0^+} \frac{\cos x}{x^2}$ (*f*) $\displaystyle\lim_{x\to\infty} [\ln(4x+9) - \ln(x+7)]$

(*g*) $\displaystyle\lim_{x\to 0^+} (3x)^x$ (*h*) $\displaystyle\lim_{x\to 0^+} (1+2x)^{1/x}$ (*i*) $\displaystyle\lim_{x\to\infty} \left(1+\frac{k}{x}\right)^x$

(*j*) $\displaystyle\lim_{x\to 0} \frac{\arctan 4x}{\sin 2x}$ (*k*) $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{3\sec x}$ (*l*) $\displaystyle\lim_{x\to 1^+} \frac{x^2+1}{1-x}$

(*m*) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x}\right)^{x^2}$

38.9 Solutions

1. Make sure to check those stages at which l'Hôpital's rule applies.

(*a*) $\lim_{x\to 0} \frac{\cos 4x - \cos 2x}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{-4\sin 4x + 2\sin 2x}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{-16\cos 4x + 4\cos 2x}{2} = \frac{-16+4}{2} = -6.$

(*b*) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{e^x - 1}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$

(*c*) $\lim_{x\to 0^+} 2x\ln x = \lim_{x\to 0^+} \frac{2\ln x}{\frac{1}{x}} \overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -\frac{2x^2}{x} = \lim_{x\to 0^+} -2x = 0.$

(*d*) $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$ (i.e., $\to \frac{2}{\infty}$)

(*e*) $\lim_{x\to 0^+} \frac{\cos x}{x^2} \to \frac{1}{0^+}$: $+\infty$. l'Hôpital's rule does not apply.

(*f*) $\lim_{x\to\infty} \ln(4x+9) - \ln(x+7) = \lim_{x\to\infty} \ln\left(\frac{4x+9}{x+7}\right) = \ln\left(\lim_{x\to\infty} \frac{4x+9}{x+7}\right) \overset{\text{l'Ho}}{=} \ln \lim_{x\to\infty} \frac{4}{1} = \ln 4.$

(*g*) Let $y = \lim_{x\to 0^+} (3x)^x$. We want to find y. Using the log process,

$$
\begin{aligned}
\ln &= \ln(\lim_{x\to 0^+} (3x)^x) \\
\ln y &= \lim_{x\to 0^+} \ln(3x)^x \\
\ln y &= \lim_{x\to 0^+} x\ln 3x \\
\ln y &= \lim_{x\to 0^+} \frac{\ln 3x}{\frac{1}{x}} \\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{\frac{3}{3x}}{-\frac{1}{x^2}} \\
\ln y &= \lim_{x\to 0^+} -x \\
\ln y &= 0.
\end{aligned}
$$

But $\ln y = 0$ implies $y = e^0 = 1$. So $\lim_{x\to 0^+} (3x)^x = y = 1$.

(*h*) "1^∞": Let $y = \lim_{x\to 0^+} (1+2x)^{1/x}$, so

$$
\begin{aligned}
\ln y &= \ln \lim_{x\to 0^+} (1+2x)^{1/x} \\
\ln y &= \lim_{x\to 0^+} \ln (1+2x)^{1/x} \\
\ln y &= \lim_{x\to 0^+} \frac{1}{x}\ln(1+2x) \\
\ln y &= \lim_{x\to 0^+} \frac{\ln(1+2x)}{x} \\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{\frac{2}{1+2x}}{1} \\
\ln y &= \frac{2}{1} \\
\ln y &= 2.
\end{aligned}
$$

But $\ln y = 2$ implies $y = e^2$. So $\lim_{x\to 0^+} (1+2x)^{1/x} = e^2$.

(*i*) "1^∞": Let $y = \lim_{x\to 0^+} (1+kx)^{1/x}$, so

$$\begin{aligned}
\ln y &= \ln \lim_{x\to 0^+} (1+kx)^{1/x} \\
\ln y &= \lim_{x\to 0^+} \ln (1+kx)^{1/x} \\
\ln y &= \lim_{x\to 0^+} \frac{1}{x}\ln(1+kx) \\
\ln y &= \lim_{x\to 0^+} \frac{\ln(1+kx)}{x} \\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{\frac{k}{1+kx}}{1} \\
\ln y &= \frac{k}{1} \\
\ln y &= k.
\end{aligned}$$

But $\ln y = k$ implies $y = e^k$. So $\lim\limits_{x\to 0^+} (1+kx)^{1/x} = e^k$.

(*j*) $\lim\limits_{x\to 0} \dfrac{\arctan 4x}{\sin 2x} \overset{\text{l'Ho}}{=} \lim\limits_{x\to 0} \dfrac{\frac{4}{1+16x^2}}{2\cos 2x} = \dfrac{\frac{4}{1}}{2} = 2.$

(*k*) $\lim\limits_{x\to 0} \dfrac{\sin 4x}{3\sec x} = \dfrac{0}{3} = 0$. l'Hôpital's rule does not apply.

(*l*) $\lim\limits_{x\to 1^+} \dfrac{x^2+1}{1-x} \to \dfrac{2}{0^-} : -\infty$. l'Hôpital's rule does not apply.

(*m*) "∞^0": Let $y = \lim\limits_{x\to 0^+} \left(\frac{1}{x}\right)^{x^2}$, so

$$\begin{aligned}
\ln y &= \ln \lim_{x\to 0^+} \left(\tfrac{1}{x}\right)^{x^2} \\
\ln y &= \lim_{x\to 0^+} \ln \left(\tfrac{1}{x}\right)^{x^2} \\
\ln y &= \lim_{x\to 0^+} x^2 \ln(\tfrac{1}{x}) \\
\ln y &= \lim_{x\to 0^+} \frac{-\ln x}{\frac{1}{x^2}} \\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{-\frac{1}{x}}{-\frac{2}{x^3}} \\
\ln y &= \lim_{x\to 0^+} \frac{x^3}{2x} \\
\ln y &= \lim_{x\to 0^+} \frac{x^2}{2} \\
\ln y &= 0.
\end{aligned}$$

But $\ln y = 0$ implies $y = 1$. So $y = \lim\limits_{x\to 0^+} \left(\frac{1}{x}\right)^{x^2} = 1$.