2.0 5-Minute Review: Polynomial Functions

Note: These 5-minute reviews will be a feature of class for the first few weeks. They cover material that you should already know. They are meant to alert you to material that that is essential to the course and that you should review if it is not familiar to you.

You should be familiar with polynomials. They are among the simplest of functions.

DEFINITION. A **polynomial** is a function of the form

$$y = p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a *non-negative integer* and each a_i is a real number (constant). If $a_n \neq 0$, then n is the **degree** of the polynomial (or highest power). The domain of a polynomial is all real numbers, $(-\infty, \infty)$.

Degree 1 polynomials: Have the form $y = a_1 x + a_0$ and are just equations of lines. The more familiar form is $y = mx + b$ (where $m = a_1$ is the slope and $b = a_0$ is the y-intercept).

Degree 2 polynomials: Have the form $y = f(x) = a_2x^2 + a_1x + a_0$ or $y = ax^2 + bx + c$. These are the familiar *quadratic functions* or *parabolas*.

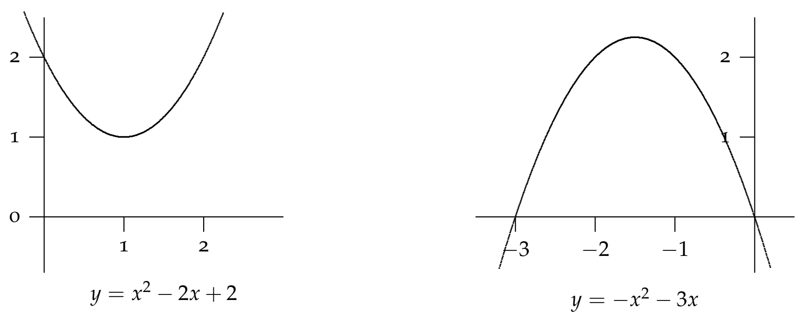


Figure 2.1: Two 'parabolas' (degree 2 polynomials). When the coefficient of x^2 is negative, the parabola is upside-down.

Degree 3 polynomials: Have the form $y = f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. Such polynomials are also called 'cubics' since the degree is 3. Such polynomials can have either 1, 2, or 3 roots.

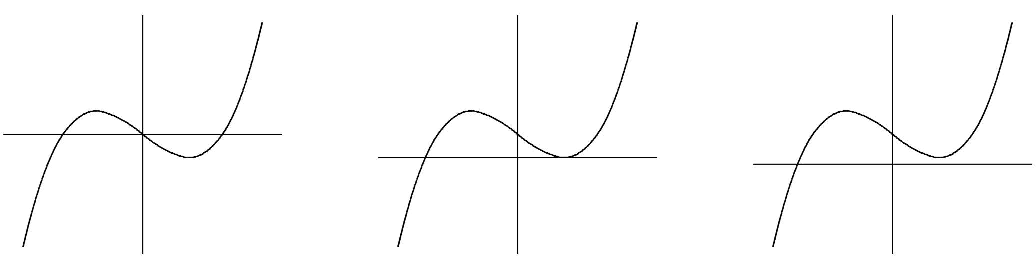


Figure 2.2: Three 'cubics' (degree 3 polynomials) with 3, 2, and 1 roots, respectively.

YOU TRY IT 2.1. Identify which functions are polynomials and determine their degree.

(*a*) $p(x) = -4x^3 + 2x + 11$	(*b*) $q(x) = \dfrac{1}{5x^2} - \dfrac{7}{x}$	(*c*) $r(t) = \dfrac{t^2+1}{t^2-1}$
(*d*) $p(x) = \sin(x^2+1)$	(*e*) $s(x) = 2x^2 - x^{1/2} + 7$	(*f*) $q(t) = \sqrt{t^3 + t^2 + 1}$
(*g*) $r(x) = 11$	(*h*) $r(x) = 3^{1/2}x^4 - 2x + \pi$	(*i*) $f(x) = -\dfrac{2}{3}x^5 + 3x^4 + x^2 - 11$
(*j*) $p(x) = 5x^2 - x^{1/3} - 23$	(*k*) $g(x) = 6x^{-2} + 4x^{-1} + 2$	(*l*) $q(x) = 3x - 4x^2 + \dfrac{x^3}{6}$

Answers to YOU TRY IT 2.1 . Polynomials (degree): a (3), g (0), h (4), i (5), l (3).

Rates of Change and the Slope Problem

2.1 Slopes and Average Rates

1. *Goal:* Recall that our goal is to define the *slope of a curve*.

2. *Key Observation:* The only slopes we know how to calculate are slopes of lines. We must somehow use the slope of a line to determine the slope of a curve. Find the line with the same slope as the curve at the point in question. This is called the **tangent line**.

3. *Set Up:* Find the slope of $y = f(x)$ at an arbitrary point $(a, f(a))$ on the curve.

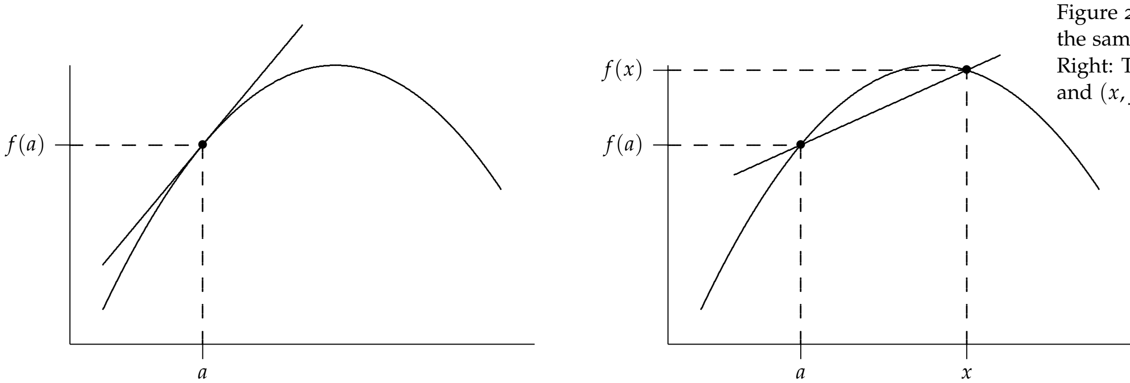


Figure 2.3: Left: The *tangent line* has the same slope as the curve at $(a, f(a))$. Right: The *secant line* through $(a, f(a))$ and $(x, f(x))$.

One of the ways we can find slopes is to use two points. We have one point $(a, f(a))$. We can get a second nearby point on the curve, call it $(x, f(x))$. Lines that pass through two distinct points of the curve are *secant lines*. The slope of the secant line through $(a, f(a))$ and $(x, f(x))$ is given by the

$$\boxed{m_{\text{sec}} = \frac{f(x) - f(a)}{x - a} = \text{difference quotient.}} \tag{2.1}$$

This makes sense as long as $x \neq a$, i.e., as long as the two points are distinct.

4. *Interpretation:* If f represents position (distance) and x represents time, then

$$\boxed{\text{difference quotient} = m_{\text{sec}} = \frac{f(x) - f(a)}{x - a} = \frac{\Delta\,\text{position}}{\Delta\,\text{time}} = \text{Average Velocity.}} \tag{2.2}$$

More generally, if $f(x)$ represents any quantity and x represents time, then the difference quotient[1] represents the average rate of change. This interpretation is

[1] The term 'difference quotient' as defined in (2.1) will be used repeated in this course. Make sure that you are comfortable with it.

what makes calculus so useful in other disciplines.

$$\boxed{\text{difference quotient} = m_{\text{sec}} = \frac{f(x) - f(a)}{x - a} = \text{Average Rate of Change.}} \tag{2.3}$$

The difference quotient can be interpreted as average velocity, or average flow rate of a stream, average acceleration, etc.

2.2 *Instantaneous Rates*

We have identified average rates of change as secant slopes of curves. But our goal is to determine the slope of a curve *right at a particular point* $(a, f(a))$ on the curve, not an average slope between two different points $(a, f(a))$ and $(x, f(x))$ as in Figure 2.3.

Figure 2.4 shows that we can obtain a better approximation to the slope of a curve by letting the second point $(x, f(x))$ approach the point $(a, f(a))$. We indicate this by using the symbol $x \to a$ ('x approaches a'). We can interpret the figure as a 'time-lapse' photo of x getting closer to a

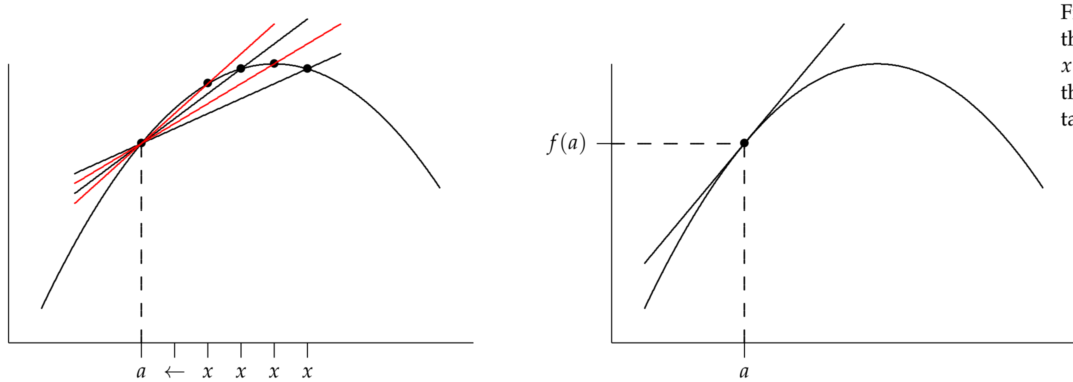


Figure 2.4: Left: The secant lines through $(a, f(a))$ and $(x, f(x))$ as $x \to a$ better approximate the slope of the curve right at $(a, f(a))$. Right: The tangent line that we seek.

Think of the curve as an icy road. The tangent slope right at the point (a, fa) is the direction that a car would travel if it slammed on its brakes right at that point and simply slid forward. The secant slopes of the lines in Figure 2.4 better approximate the tangent slope as $x \to a$. So what we want to do is let $x \to a$ in the equation for the secant slope. In other words,

$$\boxed{\text{as } x \to a,\ m_{\text{sec}} = \frac{f(x) - f(a)}{x - a} \to \text{ slope of the curve at } (a, f(a)).} \tag{2.4}$$

If $f(x)$ were position and x were time again, then

$$\boxed{\text{as } x \to a,\ \text{Average Velocity} = \frac{f(x) - f(a)}{x - a} \to \text{ Instantaneous Velocity at } (a, f(a)).} \tag{2.5}$$

But we can't just substitute $x = a$ into either of these expressions to get the slope (velocity) right at $(a, f(a))$ because we would end up with

$$\boxed{\text{when } x = a,\ \frac{f(a) - f(a)}{a - a} = \frac{0}{0} = ????.} \tag{2.6}$$

So *what does happen as* $x \to a$? The next example illustrates how this works out.

EXAMPLE 2.1. Consider the function $f(x) = x^2 - 2x$ near $a = 3$ which is graphed below. Suppose that this represents the position (in meters) of an object moving along a straight line and that x represents time. Find the instantaneous velocity of the object at $a = 3$ which is the same as finding the slope tan of this curve right at $a = 3$.

SOLUTION. We do so by calculating some average velocities (secant slopes) on the interval $[a, x] = [3, x]$ as x gets closer and closer to $a = 3$. Let's begin:

First find m_{sec} (average velocity) on the interval $[3, 5]$. From (2.2)

$$\text{Average Velocity} = m_{\text{sec}} = \frac{f(5) - f(3)}{5 - 3} = \frac{15 - 3}{2} = 6 \text{ (m/s)}. \tag{2.7}$$

The secant line through $(3, f(3))$ and $(5, f(5))$ with slope 6 is drawn in Figure 2.5. This slope is a rough approximation of the slope of the curve right at $(3, f(3))$.

Let's try for a better approximation: Choose x closer to $a = 3$. Try $x = 4$. This time

$$\text{Average Velocity} = m_{\text{sec}} = \frac{f(4) - f(3)}{4 - 3} = \frac{8 - 3}{1} = 5 \text{ (m/s)}. \tag{2.8}$$

Plot this secant line through $(3, f(3))$ and $(4, f(4))$. Does this line appear to have a slope closer to the slope of the curve right at $(3, f(3))$ than our first secant line?

Take a second and try one more point: Choose x closer still to $a = 3$. Try $x = 3.5$. Finish the calculation and plot the corresponding secant line. Fill the value in the table. What do you conclude?

$$\text{Average Velocity} = m_{\text{sec}} = \frac{f(3.5) - f(3)}{3.5 - 3} = \tag{2.9}$$

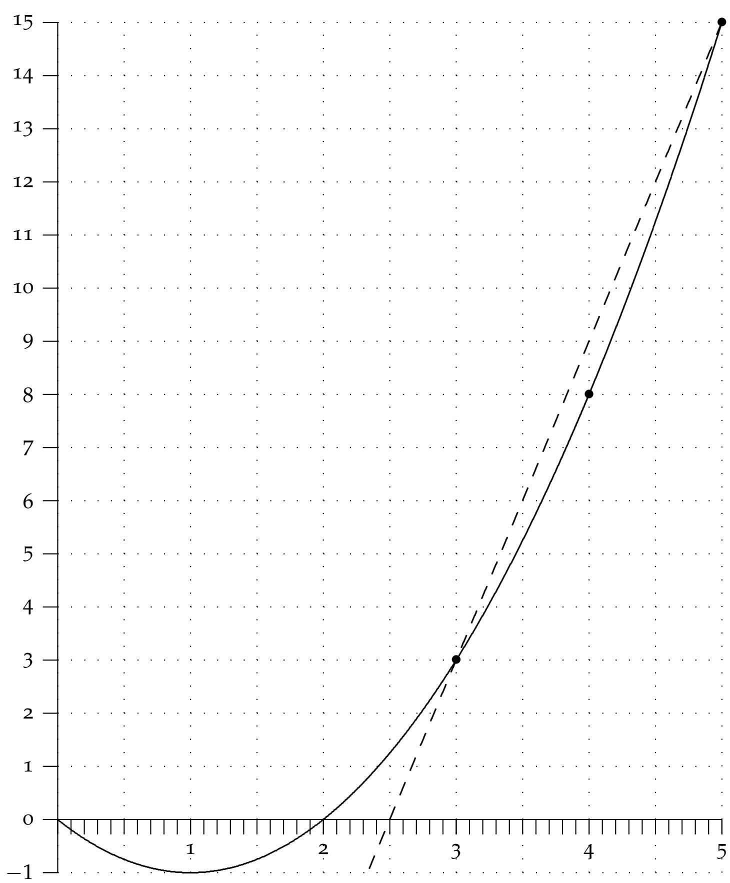


Figure 2.5: Determine the slope of $f(x) = x^2 - 2x$ at $x = 3$.

x	m_{sec} = ave vel on $[3, x]$
5	$\frac{f(5)-f(3)}{5-3} = \frac{15-3}{5-3} = 6\,\text{m/s}$
4	$5\,\text{m/s}$
3.5	
3.1	
3.01	
3.001	
3.0001	
2.9999	
2.999	
2.99	
2.9	
2.5	
2.0	

OK, we could continue in this way choosing lots of x's closer still to $a = 3$ and determine the corresponding secant slope. Instead, let's do it for a general value of x. On the inteval $[3, x]$ we find

$$\text{Ave Vel} = m_{\text{sec}} = \frac{f(x) - f(3)}{x - 3} = \frac{(x^2 - 2x) - 3}{x - 3} = \frac{(x+1)(x-3)}{x - 3} = x + 1\,\text{m/s}. \tag{2.10}$$

So for example, when $x = 5$, the secant slope would be $x+1 = 5+1 = 6$. When $x = 4$, the secant slope would be $4+1 = 5$. Or if $x = 3.5$, the secant slope would be 4.5. These match the earlier calculations above. This general calculation makes it is easy to fill in the table of values for the secant slopes. We no longer have to compute each difference quotient individually. We have a formula that gives us the slopes of the secant lines passing through $(3, f(3))$ and $(x, f(x))$, namely $m_{\text{sec}} = x+1$. When $x = 3.1$, $m_{\text{sec}} = 4.1$.

Stop! Take a moment to fill in the rest of the table. Notice that the formula works even if $x < 3$. Draw a few more secant lines in Figure 2.5.

It should be apparent from your table of values that as $x \to 3$, $m_{\text{sec}} \to 4$. Draw the line of slope 4 that passes through the point $(3, f(3))$ in Figure 2.5. This line is called the **tangent line** to $y = f(x)$ at $x = 3$. It just touches the curve once near $x = 3$. Check your drawing. Another way to say this is that as $x \to 3$, the average velocity$\to$ 4 m/s. We conclude that 4 m/s is the **instantaneous velocity** right at time 3.

One last point. We can't just let $x = 3$ in (2.2) for the slope of the secant line otherwise we'd have

$$m_{\text{sec}} = \frac{f(3) - f(3)}{3-3}$$

which is nonsensical since it involves division by 0.

More Examples

EXAMPLE 2.2 (Finding the general formula for a secant slope). Let $s(t) = -3t^2 + t + 1$.

(*a*) Find the average velocity or m_{sec} on the interval $[1, 2]$.

(*b*) Find the formula for the average velocity on the general interval $[1, t]$. Use the formula to estimate m_{sec} on the interval $[1, 1.001]$.

(*c*) Make a conjecture about m_{tan} or the instantaneous velocity right at $t = 1$.

SOLUTION. For (a)

$$\text{Ave Vel} = m_{\text{sec}} = \frac{s(2) - s(1)}{2-1} = \frac{-9 - (-1)}{2-1} = -8\,\text{m/s}.$$

For (b)

$$\text{Ave Vel} = \frac{s(t) - s(1)}{t-1} = \frac{-3t^2 + t + 1 - (-1)}{t-1} = \frac{-3t^2 + t + 2}{t-1} = \frac{(-3t-2)(t-1)}{t-1} = -3t - 2\,\text{m/s}.$$

So on the interval $[1, 1.001]$, the average velocity is $-3(1.001) - 2 = -5.003$ m/s. Looks like m_{tan} might be -5. Why?

EXAMPLE 2.3 (Using a table). Fill in the average velocities (m_{sec}) for $s(t) = 2\sin t$ on the intervals listed in the table. Then estimate m_{tan} at $t = 0$. Is factoring possible this time?

Time interval $[0, t]$	$[0, 1]$	$[0, 0.5]$	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$
Ave Vel					

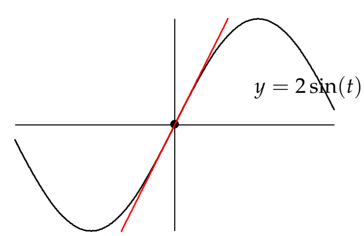


Figure 2.6: What is the slope of the red tangent line to $y = s\sin(t)$ at $t = 0$?

SOLUTION. Make sure your calculator is set in radians. We get

$$\text{Ave Vel} = \frac{2\sin(1) - 2\sin(0)}{1-0} = \frac{2\sin(1)}{1} \approx 1.6829.$$

$$\text{Ave Vel} = \frac{2\sin(0.5) - 2\sin(0)}{0.5-0} = \frac{2\sin(0.5)}{0.5} \approx 1.91770.$$

$$\text{Ave Vel} = \frac{2\sin(0.1) - 2\sin(0)}{0.1-0} = \frac{2\sin(0.1)}{0.1} \approx 1.99666.$$

$$\text{Ave Vel} = \frac{2\sin(0.01) - 2\sin(0)}{0.01-0} = \frac{2\sin(0.01)}{0.01} \approx 1.99997.$$

$$\text{Ave Vel} = \frac{2\sin(0.001) - 2\sin(0)}{0.001-0} = \frac{2\sin(0.001)}{0.001} \approx 1.9999997.$$

Looks like $m_{\text{tan}} = 2$. No factoring is possible.

Recap and a Philosophical Problem

Let's recap the important ideas we've encountered. We can interpret what we have done in a couple of different ways. In the example above,

Geometry: As $x \to 3$, the secant line $\to$ tangent line to the curve at $x = 3$. In other words, as $x \to 3$, $m_{\text{sec}} \to m_{\text{tan}} =$ the slope of $f(x)$ at $x = 3$.

Motion: As $x \to 3$, the average velocity $\to$ instantaneous velocity at $x = 3$. More generally for any quantity $f(x)$ varying with time, as $x \to a$, the average rate of change $\to$ instantaneous rate at $x = a$.

An Analogy: These observations can be combined into an analogy.

secant slope : average velocity :: tangent slope : instantaneous velocity (rate).

In particular, if $f(x) = x^2 - 2x$, then as $x \to 3$, the tangent slope $= 4 =$ slope of the curve at $x = 3$ or the instantaneous velocity is 4 m/s at $x = 3$.

2.3 Problems

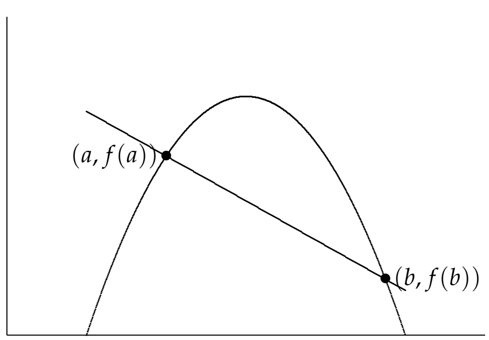
1. **Finding the general formula for a secant slope.** Let $f(x) = x^2 + x + 3$. The point $P = (4, 23)$ lies on the curve. Suppose that $Q = (x, x^2 + x + 3)$ is any other point on the graph of f. Find the slope of the secant line through P and Q. Simplify your answer.

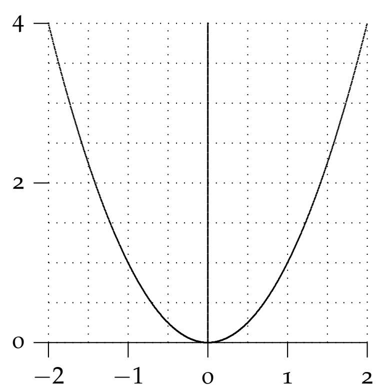
2. **Finding the general formula for a secant slope.** Let $f(x) = \frac{2}{x}$. The point $P = (\frac{1}{3}, 6)$ lies on the curve. Suppose that $Q = (x, \frac{2}{x})$ is any other point on the graph of f. Find the slope of the secant line through P and Q. Simplify your answer.

3. **Finding the general formula for a secant slope.** Let $f(x) = \sqrt{x} + 3$. The point $P = (9, 6)$ lies on the curve. Suppose that $Q = (x, \sqrt{x} + 3)$ is any other point on the graph of f. Find the slope of the secant line through P and Q. Simplify your answer. Big hint: Multiply both the numerator and denominator by $\sqrt{x} + 3$. The term $\sqrt{x} + 3$ is the **conjugate** of the original numerator in the difference quotient.

Answers: 1. $x + 5$; 2. $-\frac{6}{x}$; 3. $\frac{1}{\sqrt{x}+3}$.

4. (*a*) A **secant line** to the graph of a function $y = f(x)$ is a line that passes through two points $(a, f(a))$ and $(b, f(b))$ on the curve (see below, left). What is the general formula for the slope m_{sec} of the secant line containing the points $(a, f(a))$ and $(b, f(b))$. This general formula works for all functions.





(*b*) Suppose we want secant lines for $f(x) = x^2$. Use this particular function in your formula for m_{sec}. Simplify the expression by factoring. You now have a formula for the slope of the secant, m_{sec}, between any two points $(a, f(a))$ and $(b, f(b))$ on the curve $y = x^2$.

(*c*) Still assume that $f(x) = x^2$. Use your formula in part (b) to easily calculate the value of the slope m_{sec} if $a = -1$ and $b = 2$. This is the slope of the secant line through $(-1, f(-1))$ and $(2, f(2))$. What is the equation of the line? Draw this line on the graph (above, right). Find m_{sec} and the equation of the secant line through $(-1, f(-1))$ and $(0, f(0))$.

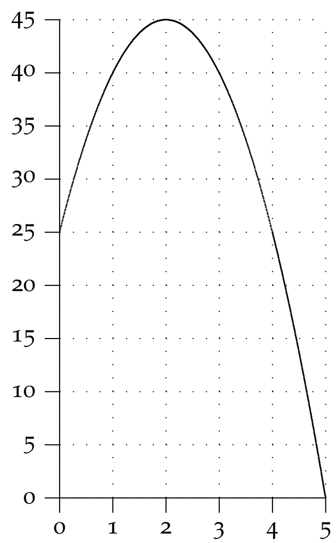
(*d*) If the secant line through $(-1, f(1))$ and $(b, f(b))$ has slope 17, what is b?

(*e*) **More interesting.** Repeat part (b) for the function $f(x) = \frac{1}{x}$. This gives a formula for m_{sec} for the curve $y = \frac{1}{x}$. Caution: Watch your algebra. Then use your formula to find the slope of secant through $(1, f(1))$ and $(2, f(2))$.

5. **Average velocity** is 'change in position over change in time.' When distance is given by a function $y = f(x)$, where x is time, then average velocity is just 'change in y over change in x' which is just the secant slope again. Assume a ball is thrown upward from the roof of Gulick Hall so that its position above the ground after x seconds is given by $y = f(x) = -5x^2 + 20x + 25$ meters over the time interval $0 \leq x \leq 5$ (see graph).

(*a*) Let $(x, f(x))$ be any point on the curve. Find the general formula for the average velocity between $(3, f(3))$ and $(x, f(x))$. Remember: This is the same thing as finding m_{sec} between these two points using 3 and x instead of a and b as in the previous problem. Simplify your answer. Hint: First factor out -5 in the numerator.

(*b*) Use the formula from part (a) to evaulate average velocities in the table that follows. This is very easy.



Time interval $[3, x]$	Average velocity on $[3, x]$
$[3, 3.1]$	
$[3, 3.01]$	
$[3, 3.001]$	
$[3, 3.0001]$	
$[3, 3]$	*!*&*%**%$***
$[2.9999, 3]$	
$[2.999, 3]$	
$[2.99, 3]$	
$[2.9, 3]$	

(*c*) Why is "*!*&*%**%$***" in the table above?

(*d*) What is the average velocity approaching as x gets close to 3?

(*e*) Determine the instantaneous velocity at $x = 3$. What is $m_{\tan}$ at $x = 3$?

6. Slope is simply the change in y over the change in x. This is easy to compute for a straight line. But in calculus we will want to determine the 'slope' of a curve $y = f(x)$. Suppose that h represents the change in x, i.e., x changes from x to $x + h$ (instead of a to b). Then $y = f(x)$ changes from $f(x)$ to $f(x+h)$. So the slope becomes

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}.$$

Care must be taken in computing and simplifying $f(x+h) - f(x)$. For example, if $f(x) = x^2 + 1$ then

$$\frac{f(x+h) - f(x)}{h} = \frac{[(x+h)^2 + 1] - [x^2 + 1]}{h} = \frac{x^2 + 2xh + h^2 + 1 - [x^2 + 1]}{h} = \frac{2xh + h^2}{h} = 2x + h.$$

Notice how it simplifies. You should be 'comfortable' with this type of calculation. Calculate $\frac{f(x+h) - f(x)}{h}$ if

(*a*) $f(x) = 3x^2 + 2$ (*b*) $f(x) = \dfrac{2}{x}$ (*c*) $f(x) = x^2 - 2x$

7. (*a*) Find the equation of the tangent line to the circle at the point indicated. Hint: How is the tangent line related to the radius?

(*b*) Circles don't have slopes in the usual sense. But how might you "define" the "slope" of the circle at the point $(3,1)$?

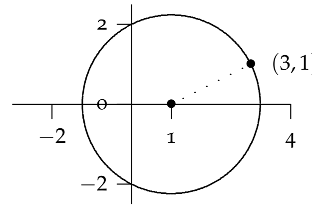


Figure 2.7: The circle for Problem 7

8. A **unit** semi-circle is drawn to the right and a general point is marked on it.

(*a*) Determine a formula for "slope" of the circle at the point indicated. (You will first need to find the y-coordinate of the point • on the circle in terms of x.)

(*b*) Use your formula to calculate the slope of the circle when $x = 0.9$ and when $x = -0.4$.

(*c*) What is the domain of your "slope" function?

(*d*) Use your formula to determine when the slope is 0. Does that make sense given the graph?

(*e*) Use your formula to determine when the slope is positive. Negative. Compare to the graph. Express your answers using interval notation.

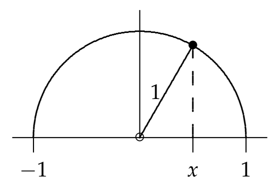


Figure 2.8: The semi-circle for Problem 8