Inverse Functions

Introduction

Often in the course of solving an equation we use one function to **undo** (the effect of) another function. We do this without really thinking about it. For example, suppose we want to solve

$$x^3 = 1000.$$

We use the cube root function to undo the cubing function

$$x = 10.$$

Implicitly what we've said is that

$$x^3 = 1000 \Rightarrow \sqrt[3]{x^3} \Rightarrow \sqrt[3]{1000} \Rightarrow x = 10.$$

The function $g(x) = \sqrt[3]{x}$ 'undid' the cubing function $f(x) = x^3$. This sounds trivial, but it is not always possible to do (or rather 'undo') this. Consider

$$\begin{aligned} x^2 &= 100 \\ \sqrt{x^2} &= \sqrt{100} \\ x &= \pm 10 \end{aligned}$$

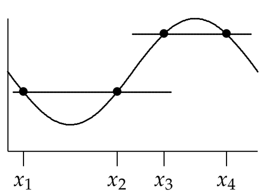
The square root function does not undo the squaring function. We can't undo the squaring function unless we know more about x... if we knew x were positive, we'd be able to say $x = 10$. So why can we undo some functions and not others?

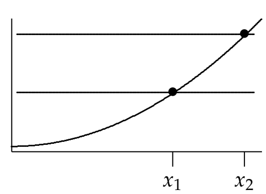
Key Idea: One-to-one Functions

DEFINITION 21.1. A function f is **one-to-one** if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. This means that f never has the same output twice.

You should memorize this definition.

EXAMPLE 21.1. The function on the left is NOT one-to-one, because $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$. Similarly $f(x_3) = f(x_4)$ even though $x_3 \neq x_4$.





The function on the right is one-to-one; it never has the same output value twice.

EXAMPLE 21.2. Show that the function $f(x) = \sqrt{x^2 + 9}$ is not one-to-one.

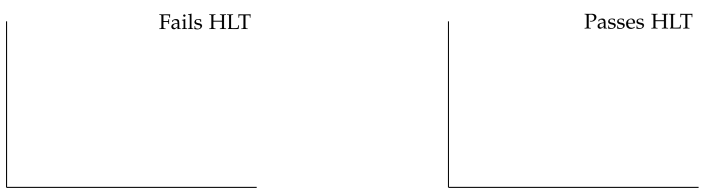
SOLUTION. Using the definition of one-to-one, we need to find two different values $x_1 \neq x_2$ so that $f(x_1) = f(x_2)$. We know that the squaring function produces the same output for inputs a and $-a$. So if we use $x_1 = 1$ and $x_2 = -1$, then $f(x_1) = \sqrt{1^2+9} = \sqrt{10}$ and $f(x_2) = \sqrt{(-1)^2+9} = \sqrt{10}$. We get the same output twice, so f is not one-to-one.

YOU TRY IT 21.1. Show that the function $f(x) = x^2 + \cos x$ is not one-to-one.

Look back at the graphs in Example 21.1. The reason that one function does not have an inverse is because a horizontal line meets the graph twice. The horizontal line represents a particular y-value. If it meets the graph twice, there must be two different x-values that have the same y-value. So we have the following theorem.

THEOREM 21.1 (Horizontal Line Test, HLT)**.** A function is one-to-one if and only if no horizontal line meets the graph more than once.

YOU TRY IT 21.2. Draw a function that passes the HLT and one that fails it.



Inverse Functions

Let's make the notion of two functions of undoing each other precise.

DEFINITION 21.2. A function g is the **inverse** of the function f if

1. $g(f(x)) = x$ for all x in the domain of f
2. $f(g(x)) = x$ for all x in the domain of g

In this situation g is denoted by f^{-1} and is called "f inverse." Notice g is the inverse of $f \iff f$ is the inverse of g.

The key fact is that

THEOREM 21.2. f has an inverse $\iff f$ is one-to-one $\iff f$ passes the HLT.

EXAMPLE 21.3. Show that $f(x) = \frac{1}{x+1}$ and $g(x) = \frac{1}{x} - 1$ are inverses.

SOLUTION. Let's check the two conditions of the definition:

1. $g(f(x)) = g\left(\dfrac{1}{x+1}\right) = \dfrac{1}{\frac{1}{x+1}} - 1 = (x+1) - 1 = x$ for all x in the domain of f
2. $f(g(x)) = f\left(\dfrac{1}{x} - 1\right) = \dfrac{1}{\left(\frac{1}{x} - 1\right) + 1} = \dfrac{1}{\frac{1}{x}} = x$ for all x in the domain of g

So we have verified the two conditions and g is f^{-1}.

The Graph of f^{-1}

Now suppose that $y = f(x)$ has an inverse, $f^{-1}(x)$ and assume that a is in the domain of f and that $f(a) = b$. Then using the definition of inverse:

$$f^{-1}(b) \overset{f(a)=b}{=} f^{-1}(f(a)) \overset{\text{Inverse}}{=} a$$

In other words

$$\boxed{f(a) = b \iff f^{-1}(b) = a}$$

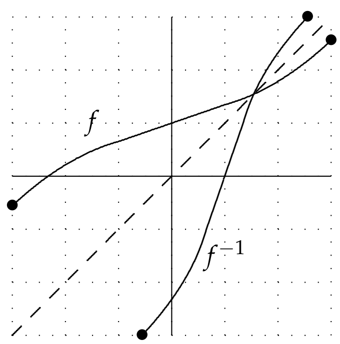
or

$$\boxed{(a,b) \text{ on the graph of } f \iff (b,a) \text{ is on the graph of } f^{-1}}$$

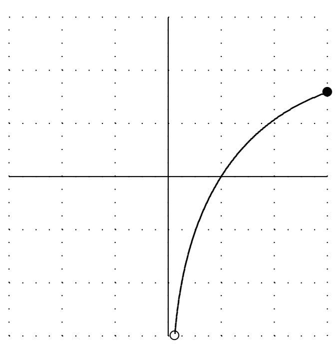
In other words, f and f^{-1} have their x and y coordinates switched. And because the x and y coordinates are switched.

- Domain of f^{-1} = Range of f
- Range of f^{-1} = Domain of f

If f is one-to-one, we can obtain the graph of f^{-1} by interchanging the x and y coordinates. If we draw the diagonal line $y = x$ and use it as a mirror, notice that the x and y axes are reflected into each other across the line. This is just another way of saying that the x and y coordinates have been switched. So to obtain the graph of f^{-1} all we need to do is to reflect the graph of f in the diagonal line $y = x$, as shown below.



YOU TRY IT 21.3. Draw the graph of f^{-1} for the function f graphed below.



When we are given the function formula for f rather than the graph, we can also find the formula for f^{-1} using the following three steps.

1. Write $y = f(x)$
2. Solve for x in terms of y. This amounts to finding $x = f^{-1}(y)$.
3. Interchange the variable names to get $y = f^{-1}(x)$.

EXAMPLE 21.4. Assume that $y = f(x) = 5x^3 + 7$ is one-to-one so that it has an inverse. To find $f^{-1}(x)$, we follow the three steps above.

1. Write $y = 5x^3 + 7$
2. Solve for x: $y = 5x^3 + 7 \Rightarrow y - 7 = 5x^3 \Rightarrow \dfrac{y-7}{5} = x^3 \Rightarrow \sqrt[3]{\dfrac{y-7}{5}} = x$

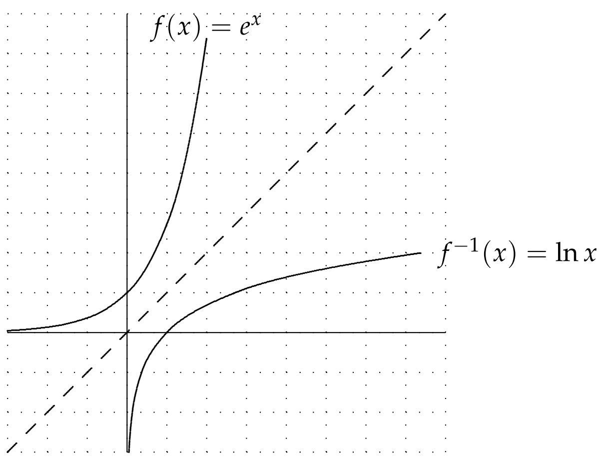
3. Interchange the variables: $f^{-1}(x) = y = \sqrt[3]{\dfrac{x-7}{5}}$

   Notice that we could also find the inverse of $y = f(x) = 5x^3 + 7$ by thinking about the order of operations of f and reversing them: To carry out f, first cube x, then multiply by 5, then add 7. To undo this, first subtract 7 to get $x - 7$, then divide by 5 to get $\frac{x-7}{5}$, and finally take the cube root $f^{-1}(x) = y = \sqrt[3]{\frac{x-7}{5}}$.

YOU TRY IT 21.4. Assume that $y = f(x) = 4 + \frac{3}{x}$ is one-to-one so that it has an inverse. Find $f^{-1}(x)$.

The Inverse of the Exponential Function e^x

Now consider $y = f(x) = e^x$. We have seen that this function is increasing and that it appears to pass the HLT, so it has an inverse. Using the graph of e^x, we can draw the graph of its inverse.



The inverse of the exponential function is the **natural log** function and is denoted by $y = \ln x$. Using the definition of inverse:

$$f(a) = b \iff f^{-1}(b) = a$$
$$e^a = b \iff \ln b = a$$

This means that logs are exponents of e (note where a is in the last line). Since inverse functions interchange the roles of x and y, this means that the domains and the ranges of the two functions are reversed.

Because logs are exponents, logs have the following very useful properties:
(1) $\ln(xy) = \ln x + \ln y$
(2) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
(3) $\ln(x^r) = r \ln x$

- Domain of $\ln x$ = Range of $e^x = (0, \infty)$ or $x > 0$.
- Range of $\ln x$ = Domain of $e^x = (-\infty, \infty)$ or all x.

Again using basic properties of inverses,

1. $f^{-1}(f(x)) = x$ or $\ln(e^x) = x$ for all x.
2. $f(f^{-1}(x)) = x$ or $e^{\ln x} = x$ for $x > 0$.

The Derivative of $y = \ln x$

As calculus students one of out first questions should be can we find the derivative of $\ln x$. So let $y = f(x) = \ln x$. Our goal is to find $\frac{dy}{dx}$ or $f'(x)$. We will use the method below a few more times to find derivatives of inverses of other functions. Assume $x > 0$ so the the natural log is defined. Start with

$$y = \ln x.$$

Take the inverse of both sides

$$e^y = e^{\ln x}.$$

Simplify

$$e^y = x.$$

Take the derivative using the chain rule (implicit differentiation)

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

Simplify

$$e^y \frac{dy}{dx} = 1$$

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

Substitute back in for y

$$\frac{dy}{dx} = \frac{1}{e^{\ln x}}$$

And simplify

$$\frac{dy}{dx} = \frac{1}{x}$$

That is, we have shown

THEOREM 21.3. The natural log function is differentiable and

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

More generally, the chain rule version is

$$\boxed{\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}.}$$

EXAMPLE 21.5. If $f(x) = \ln(x^2 + 7)$, then $u = x^2 + 7$ so

$$\frac{d}{dx}[\ln(x^2 + 7)] = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^2 + 7} \cdot 2x = \frac{2x}{x^2 + 7}.$$

If $f(x) = \ln(6x^3 \sin x)$, then since we have the log of a product, we can simplify first using a log property

$$\frac{d}{dx}[\ln(6x^3 \sin x)] = \frac{d}{dx}[\ln(6x^3) + \ln(\sin x)] = \frac{1}{6x^3} \cdot 18x^2 + \frac{1}{\sin x}\cos x = \frac{3}{x} + \cot x.$$

Similarly

$$\begin{aligned} \frac{d}{dx}\left[\ln\left(\frac{\tan 9x}{x^2+4}\right)\right] = \frac{d}{dx}[\ln(\tan 9x) - \ln(x^2 + 4)] &= \frac{1}{\tan 9x} \cdot 9\sec^2 9x + \frac{1}{x^2+4} \cdot 2x \\ &= \frac{9\sec^2 9x}{\tan 9x} + \frac{2x}{x^2+4}. \end{aligned}$$

Using the power property for logs,

$$\begin{aligned} \frac{d}{dx}\left[\ln\sqrt{x^2 + x + 10}\right] = \frac{d}{dx}\left[\tfrac{1}{2}\ln(x^2 + x + 10)\right] &= \frac{1}{2} \cdot \frac{1}{x^2 + x + 10} \cdot (2x + 1) \\ &= \frac{2x + 1}{2(x^2 + x + 10)}. \end{aligned}$$

An Important Special Case. Since we can only take logs of positive numbers, often times we use the log of an absolute value, e.g., $\ln|x|$. We can find the derivatives of such expressions as follows.

$$D_x[\ln|x|] = \begin{cases} D_x[\ln x] & \text{if } x > 0, \\ D_x[\ln(-x)] & \text{if } x < 0 \end{cases} = \begin{cases} \frac{1}{x} & \text{if } x > 0, \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases} = \frac{1}{x} \text{ if } x \neq 0$$

In other words, we get the 'same rule' as without the absolute value:

THEOREM 21.4. For $x \neq 0$,

$$D_x(\ln|x|) = \frac{1}{x}$$

The chain rule version when u is a function of x is

$$\boxed{\frac{d}{dx}(\ln|u|) = \frac{1}{u} \cdot \frac{du}{dx}.}$$

EXAMPLE 21.6. Here's one that involves a number of log properties:

$$\begin{aligned} D_t\left[\ln\left|\frac{e^t \cos t}{\sqrt{t^2+1}}\right|\right] &= D_t\left[\ln e^t + \ln|\cos t| - \ln\sqrt{t^2+1}\right| \\ &= D_t\left[t + \ln|\cos t| - \tfrac{1}{2}\ln|t^2+1|\right] \\ &= 1 + \frac{1}{\cos t} \cdot (-\sin t) - \frac{2t}{2(t^2+1)} \\ &= 1 - \tan t - \frac{t}{t^2+1} \end{aligned}$$

YOU TRY IT 21.5. Try finding these derivatives. Use log rules to simplify the functions before taking the derivative.

(*a*) $D_x\left[\ln|6x^3 \sin x|\right]$

(*b*) $D_x\left[6x^3 \ln|\sin x|\right]$ (different)

(*c*) $D_x\left[\ln\left|\frac{x^4-1}{x^2+1}\right|\right]$

(*d*) $D_t\left[\ln(t^{(e^t)})\right]$

(*e*) $D_x\left[\ln\sqrt[3]{3x^3+x+1}\right]$

(*f*) $D_s\left[\ln(5^{\ln s})\right]$