Inverse Trig Functions

Previously in Math 130...

DEFINITION 24.1. A function g is the **inverse** of the function f if

1. $g(f(x)) = x$ for all x in the domain of f
2. $f(g(x)) = x$ for all x in the domain of g

In this situation g is denoted by f^{-1} and is called "f inverse."

The Key Fact on the Existence of Inverses

THEOREM 24.1. f has an inverse $\iff$ f is one-to-one $\iff$ f passes the HLT.

The Graph of f^{-1}

Now suppose that $y = f(x)$ has an inverse, $f^{-1}(x)$ and assume that a is in the domain of f and that $f(a) = b$. Then using the definition of inverse:

$$f(a) = b \iff f^{-1}(f(a)) = f^{-1}(b) \iff a \overset{\text{Inverse}}{=} f^{-1}(b).$$

In other words

$$\boxed{f(a) = b \iff f^{-1}(b) = a}$$

or

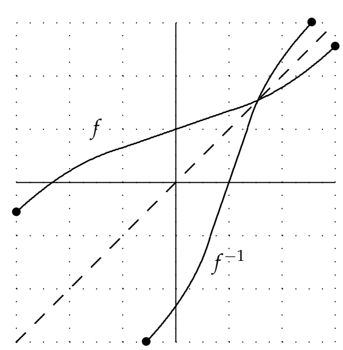
$$\boxed{(a, b) \text{ on the graph of } f \iff (b, a) \text{ is on the graph of } f^{-1}}$$

In other words, f and f^{-1} have their x and y coordinates switched. And because the x and y coordinates are switched.

- Domain of f^{-1} = Range of f
- Range of f^{-1} = Domain of f

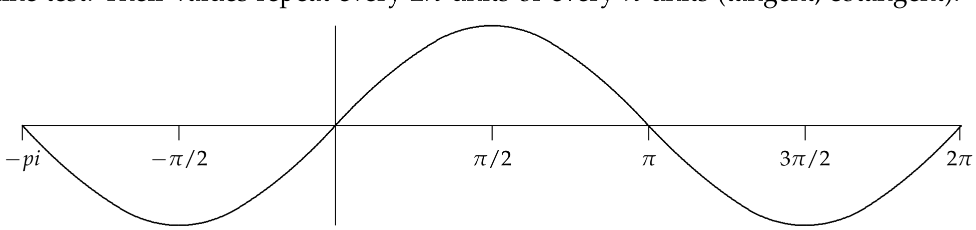
If f is one-to-one, we can obtain the graph of f^{-1} by interchanging the x and y coordinates. If we draw the diagonal line $y = x$ and use it as a mirror, notice that the x and y axes are reflected into each other across the line.

This is just another way of saying that the x and y coordinates have been switched. So to obtain the graph of f^{-1} all we need to do is to reflect the graph of f in the diagonal line $y = x$, as shown to the right.



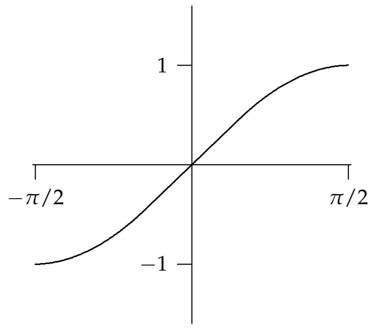
Introduction

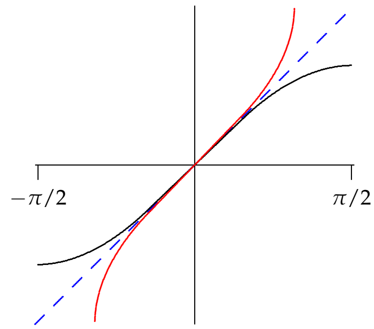
None of the trig functions have inverses because none of them pass the horizontal line test. Their values repeat every 2π units or every π units (tangent, cotangent).



The Inverse Sine Function

However, if we restrict the domain of the sine function (or any of the other trig functions) we can make the function one-to-one on the restricted interval. The figure on the left below shows $\sin x$ restricted to the interval $[-\pi/2, \pi/2]$ where it is, indeed, one-to-one (passes HLT). So it has an inverse there, which we have graphed in red the figure on the right.





The inverse sine function is denoted by $\arcsin x$. Your text uses $\sin^{-1} x$, but most students find $\arcsin x$ less confusing, and that's what we will generally use in this course. Since the domain and range of the sine and inverse sine functions are interchanged, we have

- the domain of $\arcsin x$ is the range of the restricted $\sin x$: $[-1, 1]$.

- the range of $\arcsin x$ is the domain of the restricted $\sin x$: $[-\pi/2, \pi/2]$. This is very important. It says that the output of the inverse sine function is a number (an angle) between $-\pi/2$ and $\pi/2$.

Notice since the arcsine function undoes the sine function, we get some familiar values: $\arcsin(-1) = -\pi/2$ since $\sin(-\pi/2) = -1$. Or $\arcsin(1/2) = \pi/6$ since $\sin(\pi/6) = 1/2$. Or $\arcsin(\sqrt{3}/2) = \pi/3$ since $\sin(\pi/3) = \sqrt{3}/2$.

EXAMPLE 24.1. Normally when we calculate $f^{-1}(f(x))$ we get x because the two functions undo each other. The same is true here, if the domain of $\sin x$ is appropriately restricted to $[-\pi/2, \pi/2]$. For example,

$$\arcsin(\sin(\pi/4)) = \arcsin(\sqrt{2}/2) = \pi/4.$$

But if we take a value outside of the restricted domain $[-\pi/2, \pi/2]$ of the sine function

$$\arcsin(\sin(3\pi/4)) = \arcsin(\sqrt{2}/2) = \pi/4.$$

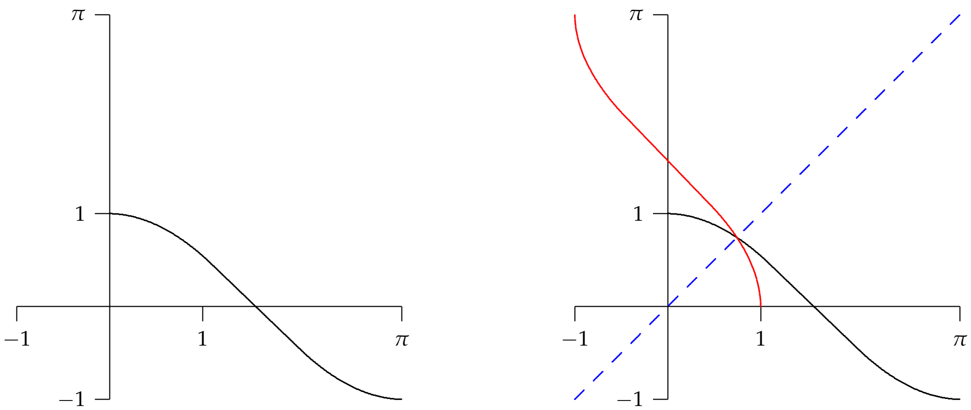
Or

$$\arcsin(\sin(3\pi)) = \arcsin(0) = 0.$$

The two functions do not undo each other since the arcsine function can only return values (or angles) between $-\pi/2$ and $\pi/2$.

The Inverse Cosine Function

We can restrict the domains of the other trig functions so that they, too, have inverses. The figure on the left below shows $\cos x$ restricted to the interval $[0, \pi]$ where it is, indeed, one-to-one. So it has an inverse there, which we have graphed in red the figure on the right.



The inverse cosine function is denoted by $\arccos x$. Since the domain and range of the cosine and inverse cosine functions are interchanged, we have

- the domain of $\arccos x$ is the range of the restricted $\cos x$: $[-1, 1]$.
- the range of $\arccos x$ is the domain of the restricted $\cos x$: $[0, \pi]$.

EXAMPLE 24.2. Again we have to be careful about calculating the composites of these inverse functions. They are only inverses when the inputs are in the correct domains. For example,

$$\arccos(\cos(\pi/4)) = \arccos(\sqrt{2}/2) = \pi/4.$$

But if we take a value outside of the restricted domain $[0, \pi]$ of the cosine function

$$\arccos(\cos(-\pi/4)) = \arccos(\sqrt{2}/2) = \pi/4.$$

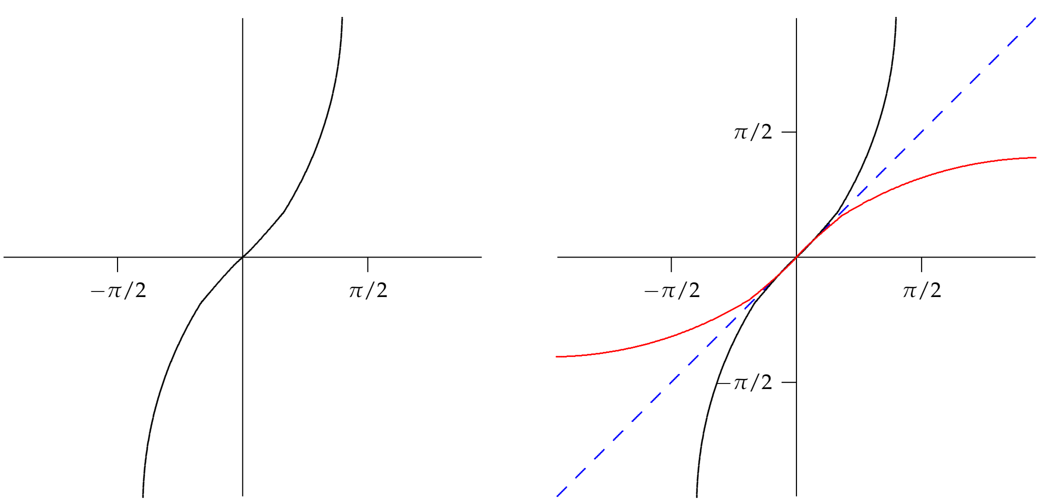
Or

$$\arccos(\cos(3\pi)) = \arccos(-1) = \pi.$$

The two functions do not always undo each other since the inverse cosine function can only return values between 0 and π.

The Inverse Tangent Function

The figure on the left below shows $\tan x$ restricted to the interval $(-\pi/2, \pi/2)$ where it is, indeed, one-to-one. So it has an inverse there, which we have graphed in red the figure on the right.



The inverse tangent function is denoted by $\arctan x$. Since the domain and range of the tangent and inverse tangent functions are interchanged, we have

- the domain of $\arctan x$ is the range of the restricted $\tan x$: $(-\infty, \infty)$.

- the range of $\arctan x$ is the domain of the restricted $\tan x$: $(-\pi/2, \pi/2)$.

EXAMPLE 24.3. Again we have to be careful about calculating the composites of these inverse functions. They are only inverses when the inputs are in the correct domains. For example,

$$\arctan(\tan(\pi/4)) = \arctan(1) = \pi/4.$$

But if we take a value outside of the restricted domain $(-\pi/2, \pi/2)$ of the tangent function

$$\arctan(\tan(3\pi/4)) = \arctan(-1) = -\pi/4.$$

Or

$$\arctan(\tan(3\pi)) = \arctan(0) = 0.$$

The two functions do not always undo each other since the inverse tangent function can only return values between $-\pi/2$ and $\pi/2$.

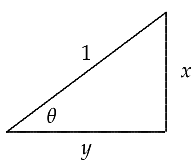
We will concentrate only on the the three inverse functions discussed above. I will leave it to you to read about the other inverse trig functions in your text.

Evaluation Using Triangles

Drawing appropriate right triangles can help evaluate complicated expressions involving the inverse trig functions.

EXAMPLE 24.4. Evaluate $\cos(\arcsin x)$.

SOLUTION. Remember that $\arcsin x = \theta$ where θ is just the angle whose sine is x. We want the cosine of this same angle. So let's draw a right triangle with angle θ whose sine is x. Since the sine function is $\frac{\text{opp}}{\text{hyp}}$ we can use the triangle below.



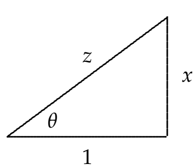
$$x^2 + y^2 = 1 \Rightarrow y = \sqrt{1-x^2}.$$

Notice $\sin\theta = \frac{x}{1} = x$. So $\arcsin x = \theta$. (θ is the angle whose sine is x.) So

$$\cos(\arcsin x) = \cos(\theta) = \frac{y}{1} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

EXAMPLE 24.5. Evaluate $\sec(\arctan x)$.

SOLUTION. This time we draw a triangle whose tangent is x.



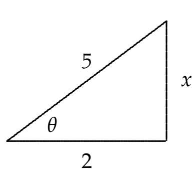
$$z^2 = 1^2 + x^2 \Rightarrow z = \sqrt{1+x^2}.$$

So

$$\sec(\arctan x) = \sec(\theta) = \frac{z}{1} = \sqrt{1+x^2}.$$

EXAMPLE 24.6. Evaluate $\sin(\arccos 2/5)$.

SOLUTION. This time we draw a triangle whose cosine is $2/5$.



$$2^2 + x^2 = 5^2 \Rightarrow x = \sqrt{5^2 - 2^2} = \sqrt{21}.$$

So

$$\sin(\arccos 2/5) = \sin(\theta) = \frac{x}{5} = \frac{\sqrt{21}}{5}.$$

YOU TRY IT 24.1. Evaluate $\sin(\arctan x)$ and $\cos(\arcsin 3/4))$.

Derivatives of $\arcsin x$ *and* $\arctan x$

Surprisingly, it is relatively easy to determine the derivatives of the inverse trig functions, assuming that they are differentiable. We will use implicit differentiation (really just the chain rule in disguise) just as we did when we figured out the derivative of $\ln x$.

Let's first determine the derivative of $y = \arcsin x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. We want to find $\frac{dy}{dx}$. First apply the inverse:

$$y = \arcsin x$$
$$\sin(y) = \sin(\arcsin x) = x.$$

Now take the derivative using implicit differentiation on the left:

$$D_x[\sin(y)] = D_x[x]$$
$$\cos(y)\frac{dy}{dx} = 1$$

Solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\cos(\arcsin x)}$$

But in Example 24.4 we found that $\cos(\arcsin x) = \sqrt{1-x^2}$ so we have

$$\frac{dy}{dx} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}.$$

That is

$$\boxed{\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}.}$$

The derivative of $y = \arctan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is determined in a similar fashion. We want to find $\frac{dy}{dx}$. First apply the inverse:

$$y = \arctan x$$
$$\tan(y) = \tan(\arctan x) = x.$$

Now take the derivative using implicit differentiation on the left:

$$D_x[\tan(y)] = D_x[x]$$
$$\sec^2(y)\frac{dy}{dx} = 1$$

Solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{\sec^2(\arctan x)}$$

But in Example 24.5 we found that $\sec(\arctan x) = \sqrt{1+x^2}$ so we have $\sec^2(\arctan x) = 1 + x^2$. Therefore

$$\frac{dy}{dx} = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1+x^2}.$$

That is

$$\boxed{\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.}$$

YOU TRY IT 24.2 (Extra Credit). Determine the formula for the derivative of $\arccos x$ using the method above. Show your work.

Keep going and find the derivatives of the remaining three inverse trig functions. Again show your work.

Chain Rule Versions

The chain rule versions of both derivative formulas are:

$$\boxed{\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}} \qquad \boxed{\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2}\frac{du}{dx}}$$

EXAMPLE 24.7. Let's use these formulas to find the derivatives of the following:

$$\frac{d}{dx}(\arctan e^{3x}) = \frac{1}{1+(e^{3x})^2} \cdot 3e^{3x} = \frac{3e^{3x}}{1+e^{6x}}. \qquad (u = e^{3x})$$

$$\frac{d}{dx}(\arcsin 3x^2) = \frac{1}{\sqrt{1-(3x^2)^2}} \cdot 6x = \frac{6x}{\sqrt{1-9x^4}}. \qquad (u = 3x^2)$$

$$\frac{d}{dx}(e^{\arctan 3x}) = e^{\arctan 3x}\frac{1}{1+9x^2} \cdot 3 = \frac{3e^{\arctan 3x}}{1+9x^2}.$$

$$\begin{aligned}\frac{d}{dx}(\sin 2x \arctan 5x^2) &= 2\cos 2x \arctan 5x^2 + \sin 2x \cdot \frac{1}{1+25x^4} \cdot 10x \\ &= 2\cos 2x \arctan 5x^2 + \frac{10x \sin 2x}{1+25x^4}.\end{aligned}$$

$$\frac{d}{dx}(\ln|\arcsin 3x|) = \frac{1}{\arcsin 3x} \cdot \frac{1}{\sqrt{1-9x^2}} \cdot 3 = \frac{3}{(\arcsin 3x)\sqrt{1-9x^2}}.$$

$$D_x(|\arcsin(\ln 3x)) = \frac{1}{\sqrt{1-[\ln(3x)]^2}} \cdot \frac{1}{3x} \cdot 3 = \frac{1}{x\sqrt{1-[\ln(3x)]^2}}. \qquad (u = \ln(3x))$$

YOU TRY IT 24.3. Find the derivatives of these functions:

$$\frac{d}{dx}\arctan(6x^2)] = \qquad \frac{d}{dx}[\arcsin(\sqrt{x})] =$$

$$\frac{d}{dx}[\arctan(e^{2x}] = \qquad \frac{d}{dx}[\arcsin(\arcsin x)] =$$

$$\frac{d}{dx}[\arctan(\ln|6x|)] = \qquad \frac{d}{dx}[\arcsin(6e^{\sin x})] =$$

$$\frac{d}{dx}(e^{2\arcsin x^2}) =$$

$$\frac{d}{dx}[(\arcsin 2x)(\tan 5x^2)] =$$

$$\frac{d}{dx}(\ln|\arctan e^{x^4+1}|) =$$

The answers are on the last page (on line) of this section.

Logarithmic Differentiation

There are still types of functions that we have not tried to differentiate yet. Sometimes we can make use of our existing techniques and clever algebra to find derivatives of very complicated functions. **Logarithmic differentiation** refers to the process of first taking the natural log of a function $y = f(x)$, then solving for the derivative $\frac{dy}{dx}$. On the surface of it, it would seem that logs would only make a complicated function *more* complicated. But remember that logs turn powers into products and products into sums. That's the key.

Let's look at the Extra Credit problem from Exam II to illustrate the idea.

EXAMPLE 24.8. Use the chain rule and implicit differentiation along with logs to find the derivative of $y = f(x) = x^x$.

SOLUTION. We begin by taking the natural log of both sides and simplifying using log properties.

$$\ln y = \ln x^x \overset{\text{Powers}}{=} x \ln x.$$

Remember we want to find $\frac{dy}{dx}$, so take the derivative of both sides (implicitly on the left).

$$\begin{aligned}\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x) \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln(x) + 1 \\ \frac{dy}{dx} &\overset{\text{Solve}}{=} y[\ln(x) + 1] \\ \frac{dy}{dx} &\overset{\text{Substitute}}{=} x^x[\ln(x) + 1]\end{aligned}$$

In other words, we have shown that $\frac{d}{dx}(x^x) = x^x[\ln(x) + 1]$. Neat! Easy!

Here are a couple more.

EXAMPLE 24.9. Find the derivative of $y = (1 + x^2)^{\tan x}$.

SOLUTION. Take the natural log of both sides and simplify using log properties.

$$\ln y = \ln(1 + x^2)^{\tan x} \overset{\text{Powers}}{=} \tan x \ln(1 + x^2).$$

Take the derivative of both sides (implicitly on the left) and solve for $\frac{dy}{dx}$.

$$\begin{aligned}\frac{d}{dx}(\ln y) &= \frac{d}{dx}\left(\tan x \ln(1 + x^2)\right) \\ \frac{1}{y} \cdot \frac{dy}{dx} &= \sec^2 x \ln(1 + x^2) + \tan x \cdot \frac{2x}{1 + x^2} \\ \frac{dy}{dx} &\overset{\text{Solve}}{=} y\left[\sec^2 x \ln(1 + x^2) + \frac{2x \tan x}{1 + x^2}\right] \\ \frac{dy}{dx} &\overset{\text{Substitute}}{=} \ln(1 + x^2)^{\tan x}\left[\sec^2 x \ln(1 + x^2) + \frac{2x \tan x}{1 + x^2}\right]\end{aligned}$$

So $\frac{d}{dx}\left(\ln(1 + x^2)^{\tan x}\right) = \ln(1 + x^2)^{\tan x}\left[\sec^2 x \ln(1 + x^2) + \frac{2x \tan x}{1 + x^2}\right]$. Not bad!

EXAMPLE 24.10. Find the derivative of $y = (\ln x)^{x^3}$.

SOLUTION. Be careful. This function is NOT the same as $\ln(x^{x^3})$ which would equal $x^3 \ln x$. Instead, take the natural log of both sides and simplify using log properties.

$$\ln y = \ln(\ln x)^{x^3} \overset{\text{Powers}}{=} x^3 \ln(\ln x).$$

Do you see the difference when compared to $\ln(x^{x^3})$

Take the derivative of both sides (implicitly on the left) and solve for $\frac{dy}{dx}$.

$$\frac{1}{y}\cdot\frac{dy}{dx} = 3x^2\ln(\ln x) + x^3\cdot\frac{1}{\ln x}\cdot\frac{1}{x}$$
$$\frac{1}{y}\cdot\frac{dy}{dx} \overset{\text{Solve}}{=} y\left[3x^2\ln(\ln x) + \frac{x^3}{x\ln x}\right]$$
$$\frac{dy}{dx} \overset{\text{Substitute}}{=} (\ln x)^{x^3}\left[3x^2\ln(\ln x) + \frac{x^3}{x\ln x}\right]$$

Logs can also be used to simplify products and quotients.

EXAMPLE 24.11. Find the derivative of $y = \dfrac{(x^2-1)^5\sqrt{1+x^2}}{x^4+4}$.

SOLUTION. Use logarithmic differentiation to avoid a complicated quotient rule derivative Take the natural log of both sides and then simplify using log properties.

$$\ln y = \ln\left(\frac{(x^2-1)^5\sqrt{1+x^2}}{x^4+4}\right)$$
$$\overset{\text{Log Prop}}{=} \ln(x^2-1)^5 + \ln(1+x^2)^{1/2} - \ln(x^4+4)$$
$$\overset{\text{Log Prop}}{=} 5\ln(x^2-1) + \frac{1}{2}\ln(1+x^2) - \ln(x^4+4).$$

Take the derivative of both sides and solve for $\dfrac{dy}{dx}$.

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{10x}{x^2-1} + \frac{x}{1+x^2} - \frac{4x^3}{x^4+4}$$
$$\frac{dy}{dx} \overset{\text{Solve}}{=} y\left[\frac{10x}{x^2-1} + \frac{x}{1+x^2} - \frac{4x^3}{x^4+4}\right]$$
$$\frac{dy}{dx} \overset{\text{Substitute}}{=} \frac{(x^2-1)^5\sqrt{1+x^2}}{x^4+4}\left[\frac{10x}{x^2-1} + \frac{x}{1+x^2} - \frac{4x^3}{x^4+4}\right]$$

That would have been a real mess to do with the quotient rule (which would also require the product rule and the chain rule).

Problems

The following questions will be on the lab tomorrow or are future WebWorK problems. Get a head start.

1. Find the derivatives of the following functions. Use logarithmic differentiation where helpful.

(*a*) $y = (\sin x)^x$ (*b*) $y = x^{\sin x}$ (*c*) $(\sin x)^{\sin x}$

(*d*) $(\arcsin x)^{x^2}$ (*e*) $\left(1+\frac{1}{x}\right)^x$

2. Find the derivatives of these functions using the derivative formula for a general exponential function that we developed before Exam II. (See Theorem 3.18 on page 194).

(*a*) $5\cdot 6^x$ (*b*) $2^x\cot x$ (*c*) $x^\pi + \pi^x$ (*d*) $x^4\cdot 4^x$

(*e*) For which values of x does $x^4\cdot 4^x$ have a horizontal tangent?

Answers.

0. Answers to YOU TRY IT 24.3 .

$$\frac{d}{dx}(\arctan(6x^2)) = \frac{1}{1+36x^4}\cdot 12x = \frac{12x}{1+36x^4} \qquad \frac{d}{dx}(\arcsin(\sqrt{x})) = \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2}\cdot x^{-1/2} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$\frac{d}{dx}(\arctan(e^{2x})) = \frac{2e^{2x}}{1+e^{4x}} \qquad \frac{d}{dx}(\arcsin(\arcsin x)) = \frac{1}{\sqrt{1-(\arcsin x)^2}}\cdot\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan(\ln|6x|)] = \frac{1}{1+(\ln|6x|)^2}\cdot\frac{1}{6x}\cdot 6 = \frac{1}{x[1+(\ln|6x|)^2]}.$$

$$\frac{d}{dx}(\arcsin(6e^{\sin x})) = \frac{1}{\sqrt{1-(6e^{\sin x})^2}}\cdot(6e^{\sin x})(\cos x) = \frac{6\cos x e^{\sin x}}{\sqrt{1-(6e^{\sin x})^2}}$$

$$\frac{d}{dx}(e^{2\arcsin x^2}) = (e^{2\arcsin x^2})\cdot 2\cdot\frac{1}{\sqrt{1-(x^2)^2}}\cdot 2x = \frac{4xe^{2\arcsin x^2}}{\sqrt{1-x^4}}$$

$$\frac{d}{dx}[\arcsin 2x(\tan 5x^2)] = \frac{2\tan 5x^2}{\sqrt{1-4x^2}} + (\arcsin 2x)10x\sec^2(5x^2)$$

$$\frac{d}{dx}(\ln|\arctan e^{x^4+1}|) = \frac{1}{\arctan e^{x^4+1}}\cdot\frac{1}{1+(e^{x^4+1})^2}\cdot e^{x^4+1}\cdot 4x^3 = \frac{4x^3e^{x^4+1}}{(\arctan e^{x^4+1})(1+e^{2x^4+2})}$$

1. (*a*) $\ln y = \ln(\sin x)^x = x\ln(\sin x) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln(\sin x) + \frac{x\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^x(\ln(\sin x) + x\cot x)$.

(*b*) $\ln y = \ln x^{\sin x} = \sin x \ln x \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \cos x\ln x + (\sin x)\frac{1}{x} \Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right)$.

(*c*) $\ln y = \ln(\sin x)^{\sin x} = \sin x\ln(\sin x) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \cos x\ln(\sin x) + (\sin x)\frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^{\sin x}\cos x[\ln(\sin x) + 1]$.

(*d*) $\ln y = \ln(\arcsin x)^{x^2} = x^2\ln(\arcsin x) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = 2x\ln(\arcsin x) + x^2\frac{1}{\arcsin x}\frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = (\arcsin x)^{x^2}\left(2x\ln(\arcsin x) + \frac{x^2}{(\arcsin x)\sqrt{1-x^2}}\right)$.

(*e*) $\ln y = \ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln\left(1+\frac{1}{x}\right) + x\cdot\frac{1}{\left(1+\frac{1}{x}\right)}\cdot\frac{-1}{x^2} \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x\left(1+\frac{1}{x}\right)} \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln\left(1+\frac{1}{x}\right) - \frac{1}{(x+1)} \Rightarrow \frac{dy}{dx} = \left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{(x+1)}\right]$.

2. (*a*) $\frac{d}{dx}[5\cdot 6^x] = 5\cdot 6^x\ln 6 = 5\ln 6(6^x)$.

(*b*) $\frac{d}{dx}[2^x\cot x] = 2^x\ln 2\cot x - 2^x\csc^2 x = 2^x[\ln 2\cot x - \csc^2 x]$.

(*c*) $\frac{d}{dx}[x^\pi + \pi^x] = \pi x^{\pi-1} + \pi^x\ln\pi$.

(*d*) $\frac{d}{dx}[x^4\cdot 4^x] = 4x^3\cdot 4^x + x^4\cdot 4^x\ln 4 = x^3\cdot 4^x[4 + x\ln 4]$.

(*e*) From the previous part, the slope is 0 when $x^3\cdot 4^x[4+x\ln 4] = 0$. Therefore $x = 0$ or $x = -\frac{4}{\ln 4}$.