

Related Rates

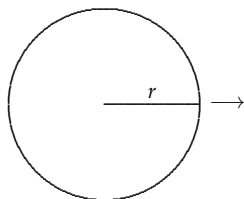
Introduction

We are familiar with a variety of mathematical or quantitative relationships, especially geometric ones. For example,

- for the sides of a right triangle we have $a^2 + b^2 = c^2$
- or the area of a circle is given by $A = \pi r^2$.

In the problems we will now discuss, one or more of these quantities will vary with time, that is, they are **implicitly** functions of time.

EXAMPLE 25.1. When you drop a pebble in a pond, a circular wave front moves outward with time.



But r is a function of time:

$$r = r(t)$$

though usually we will not write $r(t)$ explicitly. Suppose we measured and found that r was increasing at a *constant rate* of 0.2 m/s. Using proper calculus notation, we could write this as

$$\frac{dr}{dt} = 0.2.$$

Since $A = \pi r^2$, we can now ask, 'How is the area changing with respect to time?' In other words, what is

$$\frac{dA}{dt}$$

To determine the answer we use implicit differentiation.

$$\begin{aligned}\frac{d}{dt}(A) &= \frac{d}{dt}(\pi r^2) \\ \frac{dA}{dt} &= \pi 2r \frac{dr}{dt} = 2\pi r(0.2) \\ \frac{dA}{dt} &= 0.4\pi r \text{ m}^2/\text{s}\end{aligned}$$

Notice that the rate of change in the area is not constant even though the rate of change in the radius was. The change in the area depends on the particular value of r that we are interested in. For instance, when $r = 5$ m, then

$$\left. \frac{dA}{dt} \right|_{r=5} = 0.4\pi 5 = 2 \text{ m}^2/\text{s}$$

while if r were 3 m, then

$$\left. \frac{dA}{dt} \right|_{r=3} = 0.4\pi 3 = 1.2 \text{ m}^2/\text{s}$$

Differentiating Relationships

Here are some additional examples of familiar geometric relations. Assume that the quantities are varying (implicitly) with time. Let's practice finding relationships among how these quantities change.

EXAMPLE 25.2. For the sides of a right triangle: $a^2 + b^2 = c^2$ so

$$\frac{d}{dt}(a^2 + b^2) = \frac{d}{dt}(c^2) \Rightarrow 2a \frac{da}{dt} + 2b \frac{db}{dt} = 2c \frac{dc}{dt}.$$

EXAMPLE 25.3. The area of a rectangle is given by: $A = lw$ so using the product rule

$$\frac{d}{dt}(A) = \frac{d}{dt}(lw) \Rightarrow \frac{dA}{dt} = w \frac{dl}{dt} + l \frac{dw}{dt}.$$

EXAMPLE 25.4. The volume of a cylinder is given by: $V = \pi r^2 h$ so again using the product rule

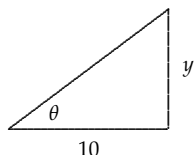
$$\frac{d}{dt}(V) = \frac{d}{dt}(\pi r^2 h) \Rightarrow \frac{dV}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}.$$

YOU TRY IT 25.1. The perimeter of a rectangle is given by: $P = 2l + 2w$. Find the relation among the rates.

YOU TRY IT 25.2. The volume of a box with a square base is given by $V = x^2 h$. Find the relation among the rates.

YOU TRY IT 25.3. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. Find the relation among the rates.

EXAMPLE 25.5. Here's a slightly different one: In the triangle below: $\tan \theta = \frac{y}{10}$ so using the chain rule



$$\frac{d}{dt}(\tan \theta) = \frac{d}{dt}\left(\frac{y}{10}\right) \Rightarrow \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{10} \frac{dy}{dt}.$$

Another way to think about this problem is that if $\tan \theta = \frac{y}{10}$, then $\theta = \arctan\left(\frac{y}{10}\right)$ so

$$\frac{d}{dt}(\theta) = \frac{d}{dt}\left[\arctan\left(\frac{y}{10}\right)\right] \Rightarrow \frac{d\theta}{dt} = \frac{\frac{1}{10}}{1 + \left(\frac{y}{10}\right)^2} \frac{dy}{dt} = \frac{1}{10 + \frac{y^2}{10}} \frac{dy}{dt} = \frac{10}{100 + y^2} \frac{dy}{dt}.$$

Related Rates Problems

In this section we will put the relationships that we have practiced differentiating into context and solve so-called 'related rates' problems.

EXAMPLE 25.6 (Classwork Example 2). In warm weather, the radius of a snowman's abdomen decreases at 2cm/hr. How fast is the volume changing when the radius of the abdomen is 80cm?

SOLUTION. Let's write in mathematical notation the rate(s) we are given and the rates we want to find.

Given Rate: $\frac{dr}{dt} = -2$ cm/hr. Notice that the rate is *negative* because the radius is *decreasing*.

Unknown Rate: $\frac{dV}{dt} \Big|_{r=80}$.

To solve the problem we need to find a relationship between the volume and the radius of a sphere and then differentiate it implicitly with respect to time, t .

Relation: $V = \frac{4}{3}\pi r^3$.

Rate-ify: To solve the problem we differentiate the relation implicitly with respect to t and then substitute in the given particular information at the last step.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

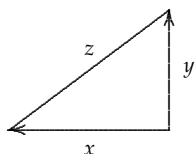
This is the general relation between the unknown and known rate.

Substitute: Now we substitute in the given information $\frac{dr}{dt} = -2$ cm/hr and the fact that we are interested in the the changing volume when $r = 80$ cm.

$$\left. \frac{dV}{dt} \right|_{r=80} = 4\pi r^2 \frac{dr}{dt} = 4\pi(80)^2(-2) = -51,200\pi \text{ cm}^3/\text{hr}.$$

EXAMPLE 25.7 (Classwork Example 1). Two students finish a conversation and walk away from each other in perpendicular directions. If one person walks at 4ft/sec and the other at 3ft/sec, how fast is the distance between the two changing at time $t = 10$ sec?

SOLUTION. This time a diagram helps:



Given Rates: $\frac{dx}{dt} = 4$ ft/s and $\frac{dy}{dt} = 3$ ft/s.

Unknown Rate: $\left. \frac{dz}{dt} \right|_{t=10}$.

Relation: To solve the problem we need to find a relationship between x , y , and z . Use the Pythagorean theorem. $z^2 = x^2 + y^2$.

Rate-ify: To solve differentiate implicitly with respect to t and then substitute in the given particular information.

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}.$$

This is the general relation between the unknown and known rates.

Substitute: Now we are interested in what is happening at time $t = 10$ s. Notice that we are NOT given the values of x , y , or z explicitly at this time. But they are easy to figure out. Remember, for constant rates, we have

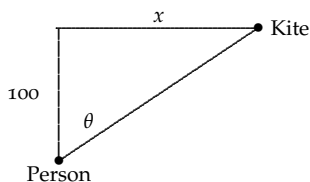
$$\text{rate} \times \text{time} = \text{distance}.$$

So when $t = 10$, we get $x = 4 \times 10 = 40$ m and $y = 3 \times 10 = 30$ m, so $z = \sqrt{40^2 + 30^2} = 50$ m. Now we substitute in this information

$$\begin{aligned} \left. z \frac{dz}{dt} \right|_{t=10} &= x \frac{dx}{dt} + y \frac{dy}{dt} \\ 50 \left. \frac{dz}{dt} \right|_{t=10} &= 40(4) + 30(3) = 250 \\ \left. \frac{dz}{dt} \right|_{t=10} &= \frac{250}{50} = 5 \text{ m/s}. \end{aligned}$$

EXAMPLE 25.8 (Classwork Example 3). A kite 100 feet above the ground moves horizontally at a rate of 8ft/s. At what rate is the angle between the string and the vertical

direction changing when 200 ft of string have been let out?



SOLUTION. Let x denote the (horizontal) side opposite θ and let z be the hypotenuse. (Mark both now.)

Given Rates: $\frac{dx}{dt} = 8$ ft/s.

Unknown Rate: $\frac{d\theta}{dt} \big|_{z=200}$.

Relation: Use trig to relate θ and x : $\tan \theta = \frac{x}{100}$.

Rate-ify: Differentiate implicitly with respect to time to obtain

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{100} \frac{dx}{dt}.$$

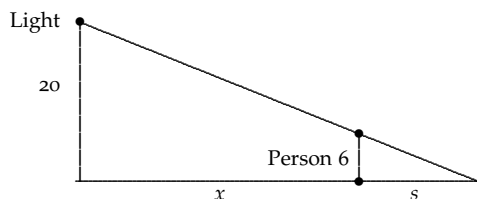
This means

$$\frac{d\theta}{dt} = \frac{1}{\sec^2 \theta} \frac{1}{100} \frac{dx}{dt} = \cos^2 \theta \frac{1}{100} \frac{dx}{dt}.$$

Substitute: Now substitute in the known values. From the triangle, when $z = 200$, $\cos \theta = \frac{100}{200} = \frac{1}{2}$, so

$$\frac{d\theta}{dt} \bigg|_{z=200} = \left(\frac{1}{2}\right)^2 \frac{1}{100} (8) = \frac{2}{100} = \frac{1}{50} \text{ rad/s.}$$

EXAMPLE 25.9 (Classwork Example 4). As you walk away from a 20 foot high lamppost, the length of your shadow changes. If you are 6 feet tall and walking at 3ft/sec, at what rate is the length of your shadow changing?



SOLUTION. Let s denote the length of the shadow and let x denote the distance from the lamp.

Given Rates: $\frac{dx}{dt} = 3$ ft/s.

Unknown Rate: $\frac{ds}{dt}$. Note in this problem we want to find this rate in general.

Relation: Use similar triangles to relate s and x .

$\frac{s}{6} = \frac{x+s}{20}$. It makes sense to solve for s in terms of x . Cross-multiply and then simplify to get:

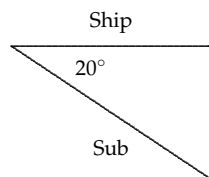
$$s = \frac{6}{14}x.$$

Rate-ify and Substitute: Differentiate implicitly with respect to time to obtain

$$\frac{ds}{dt} = \frac{6}{14} \frac{dx}{dt} = \frac{6}{14} (3) = \frac{9}{7} \text{ ft/s.}$$

Notice that the rate of change is constant and does not depend on where the person is.

EXAMPLE 25.10 (Classwork Example 5). A surface ship is moving in a straight line at 10 km/hr. An enemy sub maintains a position directly below the ship while diving at an angle of 20° to the surface. How fast is the sub moving?



SOLUTION. Let x denote the position of the surface ship (horizontal side) let z be the hypotenuse. (Mark both now.)

Given Rates: $\frac{dx}{dt} = 10$ km/hr.

Unknown Rate: $\frac{dz}{dt}$.

Relation: Use trig to relate z and x . $\cos 20^\circ = \frac{x}{z}$ or $z = \frac{1}{\cos 20^\circ} x = (\sec 20^\circ)x$.

Note: $\sec 20^\circ$ is constant.

Rate-ify and Substitute: Differentiate implicitly with respect to time to obtain

$$\frac{dz}{dt} = \sec 20^\circ \frac{dx}{dt} = \sec 20^\circ (10) \approx 10.64 \text{ km/hr.}$$

YOU TRY IT 25.4. How fast is the depth of the sub changing in the previous example?