More Related Rates

We have seen that many related rate problems involve implicit derivatives of familiar geometric relations. Before reading the examples below, try finding the derivatives with respect to time t of these relations. The answers are at the end of this section.

$$(a)\ V = \frac{4}{3}\pi r^3 \qquad (b)\ V = \pi r^2 h \qquad (c)\ S = 2x^2 + 4xh \qquad (d)\ \tan\theta = \frac{y}{5}$$

Examples

We finish our discussion of related rate problems today with a few more examples.

EXAMPLE 26.1 (Practice Problem 7). An above ground pool with a radius of 10 feet is being filled with water. If the water level is rising at a rate of 0.2 ft/min, how fast is the volume of the the water in the pool changing? (Draw and label a diagram.) [Now try page 217 #36.]

SOLUTION. Let V denote the volume and h the height of the water. The radius is constant and equal to 5.

Given Rates: $\frac{dh}{dt} = 0.2$ ft/min.

Unknown Rate: $\frac{dV}{dt}$. To solve the problem we need to find a relationship between the volume and height of the water and then differentiate it implicitly with respect to time, t.

Relation: $V = \pi r^2 h = \pi 5^2 h = 25\pi h$, since $r = 5$ is constant.

Rate-ify: Now differentiate the relation implicitly with respect to t.

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt}.$$

This is the general relation between the unknown and known rates.

Substitute: Now we substitute in the given information.

$$\frac{dV}{dt} = 25\pi(0.2) = 5\pi \text{ ft}^3/\text{min}.$$

EXAMPLE 26.2 (Practice Problem 8). An ice block (draw and label a diagram) with a square base is melting in the sun at a steady rate of 48 cm^3/hr. If the height is decreasing at a rate of 0.5 cm/hr, how fast is the edge of the base changing when the block is 10 cm in height and has an edge length of 6 cm? How fast is the surface area of the block changing at the same moment?

SOLUTION. Let x denote the edge length of the base and h the height.

Given Rates: $\frac{dV}{dt} = -48\ \text{cm}^3/\text{hr}$ and $\frac{dh}{dt} = -0.5$ cm/hr. Notice that the rates are *negative* because the quantities are *decreasing*.

Unknown Rate: $\frac{dx}{dt}\Big|_{h=80,\ x=6}$. Also if S is the surface area, we want $\frac{dS}{dt}\Big|_{h=80,\ x=6}$.

To solve the problem we need to find a relationship between the volume, height and the edge of a block and then differentiate it implicitly with respect to time, t.

Relation: $V = x^2h$, since the base is square.

Rate-ify: Now differentiate the relation implicitly with respect to t.

$$\frac{dV}{dt} = 2xh\frac{dx}{dt} + x^2\frac{dh}{dt}.$$

This is the general relation between the unknown and known rates.

Substitute: Now we substitute in the given information when $h = 10$ and $x = 6$ cm.

$$-48 = 2(6)(10)4\frac{dx}{dt} + (6)^2(-0.5) \Rightarrow -30 = 120\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -0.25 \text{ cm/hr}.$$

Relation: For the surface area problem, the four sides are rectangles with area xh and the top and bottom are squares with area x^2. So $S = 2x^2 + 4xh$.

Rate-ify: Differentiate

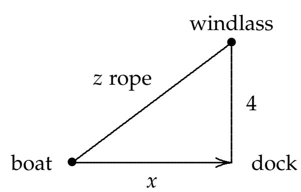
$$\frac{dS}{dt} = 4x\frac{dx}{dt} + 4h\frac{dx}{dt} + 4x\frac{dh}{dt}.$$

Substitute We know all of the rates on the right side of the equation. So substituting in we find:

$$\frac{dS}{dt} = 4(6)(-0.25) + 4(10)(-0.25) + 4(6)(-0.5) = -22 \text{ cm}^2/\text{hr}.$$

YOU TRY IT 26.1 (Practice Problem). During an animation sequence, a rectangle on a computer screen changes size. The width is increasing at 2 cm/s while the height is decreasing at 1.5 cm/s. How fast is the area of the rectangle changing when the width is 10 cm and the height is 15 cm? Is the area decreasing or increasing?

EXAMPLE 26.3. A boat floating several feet away from a dock is pulled in by a rope that is being wound up by a windlass at the rate of 3 ft/s. The windlass is 4 ft above the level of the boat. How fast is the boat moving towards the dock when the boat is 12 feet from the dock?



SOLUTION. Let x denote the distance of the boat from the dock and z the length of the rope.

Given Rates: $\frac{dz}{dt} = -3$ ft/s. Remember z is the rope.

Unknown Rate: $\frac{dx}{dt}\big|_{x=12}$.

Relation: The relationship between x and z is the Pythagorean theorem: $z^2 = x^2 + 4^2$.

Rate-ify: Now differentiate the relation implicitly with respect to t. (Remember 4 is constant.)

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} \Rightarrow z\frac{dz}{dt} = x\frac{dx}{dt}.$$

This is the general relation between the unknown and known rates.

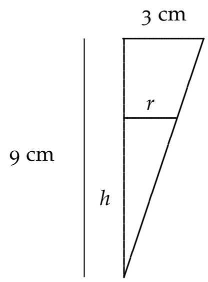
Substitute Now we substitute in the given information when $x = 12$ ft. We also need to know z. But the relation

$$z^2 = x^2 + 4^2 \Rightarrow z^2 = 12^2 + 4^2 = 160 \Rightarrow z = \sqrt{160} = 4\sqrt{10}.$$

So the rate relation becomes and $x = 6$ cm.

$$(4\sqrt{10})\frac{dz}{dt} = 12(-3) \Rightarrow \frac{dz}{dt} = -\frac{36}{4\sqrt{10}} = -\frac{9}{\sqrt{10}} \text{ ft/s}.$$

EXAMPLE 26.4 (Practice Problem 9)**.** A conical cup is 6 cm across the top and 9 cm deep. It is filled with ginger ale at a rate of 3 cc/s. A bug originally at the bottom of the cup floats on the surface of the liquid as it rises. Find the bugs "velocity" as a function of the depth h of the ginger ale. What is this rate when $h = 6$? The cross-section of the cone below may help.



SOLUTION. Let h denote the height of the liquid in the cone and r the radius at the top of the liquid.

Given Rate: $\frac{dV}{dt} = 3$ cc.

Unknown Rate: $\left.\frac{dh}{dt}\right|_{h=6}$.

Relation: The relationship between V and h is given by the volume formula for a cone: $V = \frac{1}{3}\pi r^2 h$. The hint suggests to eliminate r which we can do using similar triangles.

$$\frac{r}{h} = \frac{3}{9} \Rightarrow r = \frac{1}{3}h.$$

$$V = \tfrac{1}{3}\pi r^2 h = V = \tfrac{1}{3}\pi\left(\tfrac{1}{3}h\right)^2 h = \tfrac{\pi}{27}h^3.$$

Rate-ify: Now differentiate the relation implicitly with respect to t.

$$\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}.$$

This is the general relation between the unknown and known rates.

Substitute Now we substitute in the given information when $h = 6$ ft.

$$3 = \frac{\pi}{9}(6)^2\left.\frac{dh}{dt}\right|_{h=6} \Rightarrow \left.\frac{dh}{dt}\right|_{h=6} = \frac{27}{36\pi} = \frac{3}{4\pi}\ \text{cm/s}.$$

YOU TRY IT 26.2 (Practice Problem 10)**.** Suppose that a toy rocket is launched vertically. When it is 40 meters in the air its velocity is 10 m/sec. Assume you are laying on the ground 30 meters from the launch site. What is the rate of change in the angle of elevation of your head to track the rocket at this instant? Hint: See Classwork Example 3 from Day 26.

Answers

$(a)\ \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$ $\qquad (b)\ \dfrac{dV}{dt} = 2\pi rh\dfrac{dr}{dt} + \pi r^2\dfrac{dh}{dt}$

$(c)\ \dfrac{dS}{dt} = 4x\dfrac{dx}{dt} + 4h\dfrac{dx}{dt} + 4x\dfrac{dh}{dt}$ $\qquad (d)\ \sec^2\theta\dfrac{d\theta}{dt} = \dfrac{1}{5}\dfrac{dy}{dt}$