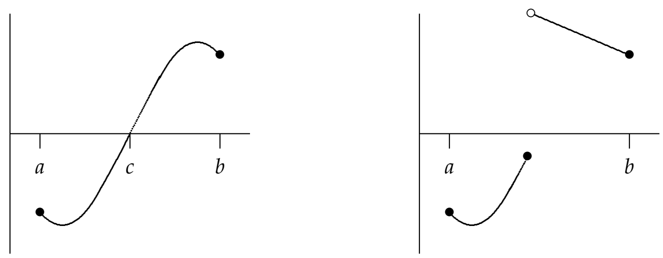
Optimization and Calculus

The IVT: An Important Property of Continuous Functions

While we know continuous functions are nice because they make evaluating limits easy, we saw earlier in the term that they posses another nice property. On an intuitive level, the graph of a continuous function f is an "unbroken" curve. If f is continuous on the closed interval $[a, b]$ with $f(a) < 0$ and $f(b) > 0$, geometric intuition tells us that there is at least one point c between a and b such that $f(c) = 0$. When f is not continuous, it is evident that such a point c need not exist.



There is nothing special about 0; the same is true for any number between $f(a)$ and $f(b)$. The first important property of continuous functions in this section is

THEOREM 28.1 (IVT: The Intermediate Value Theorem)**.** Let f be a continuous function on a closed interval $[a, b]$. Let L be any number between $f(a)$ and $f(b)$. Then there is some point c in (a, b) so that $f(c) = L$.

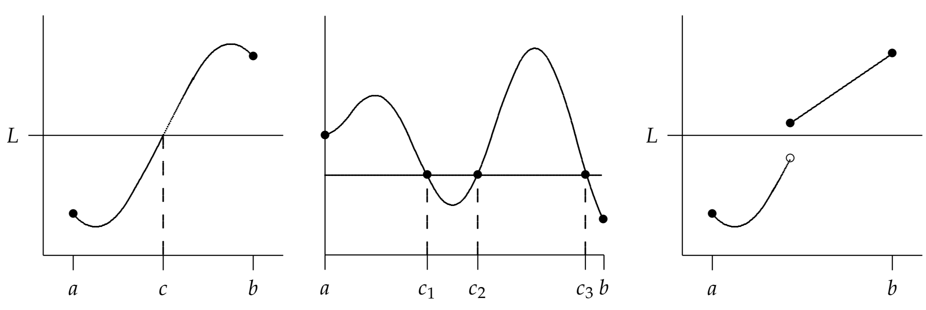


Figure 28.1: Left and middle: If f is continuous on the closed interval $[a, b]$ with L between $f(a)$ and $f(b)$, then there is at least one point c between a and b such that $f(c) = L$. Right: When f is not continuous, such a point c need not exist.

Though this theorem may seem 'obvious', the proof is surprisingly difficult and is covered in Math 331. Nonetheless, you can see how the hypothesis that f is continuous is critical. See the right-hand graph in Figure 28.1. When f is not continuous, the curve can 'jump over' the value L so that there is no point c in (a, b) with $f(c) = L$.

EXAMPLE 28.1. Prove that $p(x) = 6x^4 + 4x^3 - 2x^2 - x - 3$ has a root in the interval $[-1, 1]$.

SOLUTION. We want to solve $p(x) = 0$. Well, we probably are not going to be able to factor this polynomial. So let's see how we can apply the IVT, Theorem 28.1. Since

p is a polynomial, it is continuous on the interval $[-2, 0]$. Moreover, at the endpoints $p(-1) = 6 - 4 - 2 + 1 - 3 = -2$ and $p(1) = 6 + 4 - 2 - 1 - 3 = 4$. Notice that 0 is between $p(-1) = -2$ and $p(1) = 4$. So by the IVT, there is some number c in $(-1, 1)$ so that $p(c) = 0$. We don't know the value of c, just that it exists. Neat!! (This is why theorems like the IVT are sometimes called **existence theorems**.)

EXAMPLE 28.2. Prove that $f(x) = x^3 - 4x + \cos(\pi x)$ has a root in the interval $[0, 1]$.

SOLUTION. We want to solve $f(x) = 0$. Well, we can't factor this function. But we can apply the IVT, Theorem 28.1. First, $\cos(\pi x)$ is a composite of trig function and a polynomial, πx. Next, $x^3 - 4x$ is a polynomial and so is continuous. So f is the sum of continuous functions and so it is continuous on the interval $[0, 1]$. Moreover, at the endpoints $f(0) = 0 - 0 + 1 =$ and $f(1) = 1 - 4 - 1 = -4$. Notice that 0 is between $f(0) = 1$ and $f(1) = -4$. So by the IVT, there is some number c in $(0, 1)$ so that $f(c) = 0$. Again, we don't know the value of c, just that it exists!!

YOU TRY IT 28.1 (Hand in for extra credit). Show that $p(x) = x^4 - x^3 + x^2 + x - 1$ has at least two roots in $[-1, 1]$. Big hint: Split $[-1, 1]$ in half into two smaller closed intervals. Show that p has a root in each of the two smaller intervals.

EXAMPLE 28.3. Your parents invest \$20,000 in a savings account for you when you are 8-years-old. They want it to be worth \$50,000 ten years later when when you start HWS. If the account has an annual interest rate r, with monthly compounding, Then the amount in the account after 10 years (120 months) is

$$A(r) = 20,000\left(1 + \frac{r}{12}\right)^{120}.$$

Use the IVT to show that there is a value r in $(0, 0.10)$—i.e., an interest rate between 0 and 10% so that $A(r) = 50,000$.

SOLUTION. The function $A(r) = 20,000\left(1 + \frac{r}{12}\right)^{120}$ is a composition of continuous functions (or a polynomial of degree 120 if you multiply it out!) At any rate, it is as continuous function for all values of r. So we can apply the IVT, Theorem 28.1 on $[0, 0.10]$. $A(0) = 20,000$—the bank is paying no interest. $A(0.10) = 54140.83$. So we have

$$A(0) < 50000 < A(0.10)$$

so by the IVT, there is a value of r in $(0, 0.10)$ so that $A(r) = 50,000$.

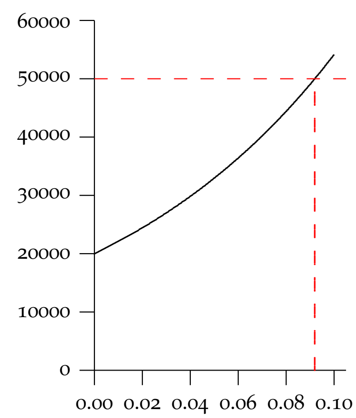
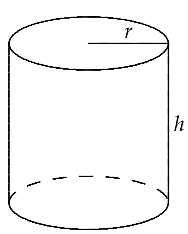


Figure 28.2: The function $A(r)$ on the interval $[0, 0.10]$. By the IVT, it must meet the horizontal line representing \$50,000 at some point in the interval.

Introduction to Extrema and Optimization

Our goal for the next several days is to apply calculus theory to the practical problem of solving optimization problems. This is one of the most important applications of elementary differential calculus and it is widely used.

EXAMPLE 28.4. A soup can needs to be designed to hold 18 oz. What dimensions for the radius and height of the can *minimize* the cost (i.e., the materials) for the can. Here, **optimal** means minimize.



EXAMPLE 28.5. Your Favorite Band is playing at the Smith Opera House. What ticket price will maximize their revenue? Here, **optimal** means maximize.

EXAMPLE 28.6. For several years I served on a committee that investigated what tuition will maximize the revenue for Hobart & William Smith Colleges.

In all of these situations, we wish to locate the maximum or minimum values of a function *efficiently* using differential calculus. Such points are called **extreme values** of the function.

Extrema Terminology

There are several types of extreme values and it is important that we define them carefully.

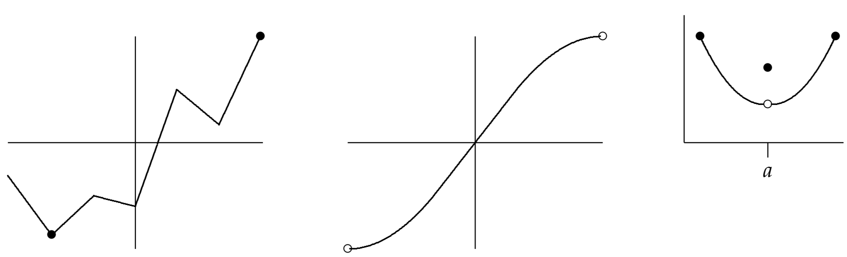
DEFINITION 28.1. Let f be a function defined on an interval I containing the point c.

(*a*) f has an **absolute (global) maximum** at c if $f(c) \geq f(x)$ for all x in I. The number $f(c)$ is the **maximum value** of f.

(*b*) f has an **absolute (global) minimum** at c if $f(c) \leq f(x)$ for all x in I. The number $f(c)$ is the **minimum value** of f.

(*c*) Absolute maxima or minima are called **absolute extreme values (extrema)** of f

The absolute extrema are marked with • in these graphs. Notice that the second graph has no absolute max or min (since it does not contain the endpoints) and that the third graph has no absolute min (the small dot is part of the graph but is not a min) and two maxima (at the endpoints.



DEFINITION 28.2. Note open intervals!

(*a*) If $f(c) \geq f(x)$ for all x in some open interval containing c, then $f(c)$ is a **relative (local) maximum** value of f. (Or: f has a local max at c.)

(*b*) If $f(c) \leq f(x)$ for all x in some open interval containing c, then f is a **relative (local) minimum** value of f. (Or: f has a local min at c.)

In the graphs above, the first has three local minima and two local maxima (note that the endpoints are not local extrema because there is no *open* interval containing the endpoints. The second and third graphs have no local extrema.

We can see in third graph that the function would have both a local and global minimum value if the removable discontinuity were eliminated, that is, if it were continuous. So we see that continuity is important in finding extreme values. On the other hand, the second graph is continuous but does not have any relative or global extrema. This time the problem is that the interval over which it is defined is not closed. If the interval were closed and the endpoints of the graph included, then it would have at least had a global max and a global min, though it still would not have had any local extrema.

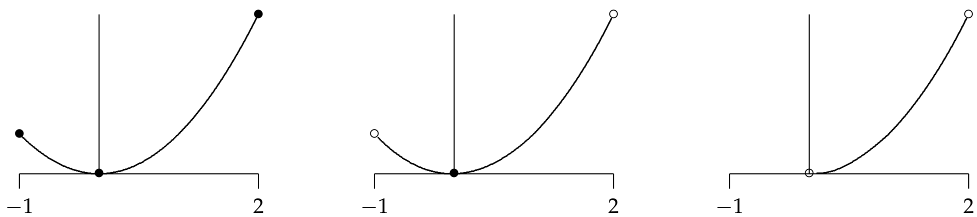
The second fact about continuous functions on closed intervals is that they always have global extrema (though they may not have local ones).

The interval matters

EXAMPLE 28.0.1. Left: The function $y = f(x) = x^2$ on the **closed interval** $[-1, 2]$ has both an absolute max and min.

Middle: The function $y = f(x) = x^2$ on the **open interval** $(-1, 2)$ has an absolute min but no absolute max.

Right: The function $y = f(x) = x^2$ on the **open interval** $(-0, 2)$ has no absolute min or max.



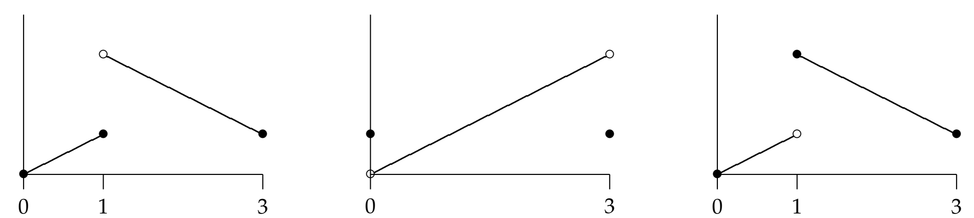
The type of function matters

The example above shows that a continuous function on an non-closed interval may not have an absolute max or min. When the interval is closed, if the function is not continuous, it may still not have have both an absolute max or min.

EXAMPLE 28.0.2. Left: A **discontinuous function** $y = f(x)$ on the **closed interval** $[0, 3]$ that has an absolute max but no absolute min. Notice f is defined at each point in the interval.

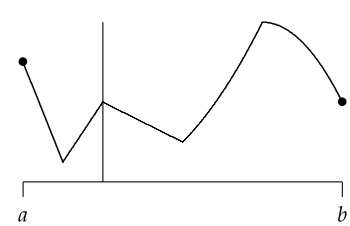
Middle: A **discontinuous function** $y = f(x)$ on the **closed interval** $[0, 3]$ that has an absolute max but no absolute min. Notice f is defined at each point in the interval.

Right: A **discontinuous function** $y = f(x)$ on the **closed interval** $[0, 3]$ that has both an absolute max and absolute min. Notice f is defined at each point in the interval.



It turns out that when we have a continuous function on a closed interval, life is good!

THEOREM 28.2 (EVT: The Extreme Value Theorem)**.** A function f that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value in that interval $[a, b]$.



YOU TRY IT 28.2. Mark the global maximum and minimum values of the continuous function on the closed interval $[a, b]$ on the figure in the margin.

YOU TRY IT 28.3. (From class). Draw a function that satisfies the given conditions or explain why this is impossible.

(*a*) A continuous function on $(1, 8)$ which has no absolute minimum

(*b*) A function which is continuous on $[1, 8]$ which has no absolute extreme points.

(*c*) A function on $[1, 8]$ which has no absolute maximum.

(*d*) A continuous function on $[1, 8]$ for which $f(1) = -3$ and $f(8) = 4$ and which is never 0 (has no roots).

(*e*) A continuous function on $(1, 8)$ for which $f(3)$ is a relative max and $f(5)$ is a relative min but for which f has no absolute max or min.

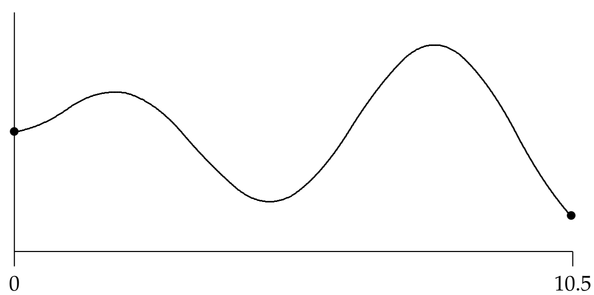
YOU TRY IT 28.4. (More general). Draw your own graph that shows that the conclusion of the EVT can fail—that is, the function may not have a global max or a global min—if the function is not continuous even if it is defined at every point on a closed interval.

Draw another function hat shows that the conclusion of the EVT can fail on a non-closed interval even if the function is continuous.

These examples show that for a function to always have both a global max and a global min, it must be continuous on a closed interval. Both conditions are important.

The IVT and EVT are two great theorems (and both are very hard to prove; take Math 331). The latter, in particular, tells us that can always optimize a continuous function on a closed interval, which is what motivated us in the first place. HOWEVER, the EVT does not tell us HOW to find the extrema, it only tells that they exist. When a function is *differentiable* on a closed interval $[a, b]$, we will see that we can give an algorithm (recipe) for how to find the extrema.

EXAMPLE 28.0.3. The function below is differentiable, hence continuous on the closed interval $[0, 10.5]$. The EVT says the function should have both an absolute max and an absolute min. Mark them. For a differentiable function where should we look for extreme values? Mark at all the local extrema. What is the common property that all these local extrema share?



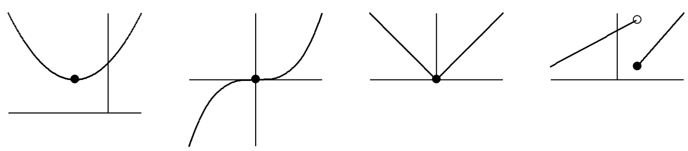
29
Optimization: Differentiability and Extrema

Our first theorem tells us that when f is differentiable, then at a local high or low point, the function has a horizontal tangent. In fact, if we first define a new term, then we can say even more.

DEFINITION 29.1. Assume that f is defined at c. Then c is a **critical number of** f if either

1. $f'(c) = 0$ or
2. $f'(c)$ does not exist.

EXAMPLE 29.1. Find the critical points of these four graphs



EXAMPLE 29.2. Find the critical numbers of $f(x) = 3x^{2/3} - 2x$.

SOLUTION. We need to see where $f'(x) = 0$ or DNE.

$$f'(x) = 2x^{-1/3} - 2 = \frac{2}{x^{1/3}} - 2 = 0 \iff \frac{1}{x^{1/3}} = 1 \iff 1 = x^{1/3} \iff 1 = x.$$

Also notice

$$f'(x) = 2x^{-1/3} - 2 = \frac{2}{x^{1/3}} - 2 \text{ DNE} \iff x = 0.$$

So f has two critical points, $x = 0, 1$.

EXAMPLE 29.3. Find the critical numbers of $f(x) = \arctan(e^{2x} - 4x)$.

SOLUTION. We need to see where $f'(x) = 0$ or DNE.

$$f'(x) = \frac{2e^{2x} - 4}{1 + (e^{2x} - 4x)^2} = 0 \iff 2e^{2x} - 4 = 0 \iff e^{2x} = 2 \iff 2x = \ln 2 \iff x = \frac{\ln 2}{2}.$$

Notice $f'(x)$ always exists, so there is only one critical number: $x = \frac{\ln 2}{2}$.

From the graphs we saw that sometimes we have extrema at critical points. The next theorem makes the connection explicit.

THEOREM 29.1 (CNT: The Critical Number Theorem)**.** If f has a local max or min at c, then c is a critical number of f.

Proof. We will examine the case where f has a a local max at c. A similar proof works for a local min. There are two possibilities: Either f is not differentiable at c or it is.

1. If f is not differentiable at c, then c is automatically a critical number by definition.

2. If f is differentiable, then we need to show $f'(c) = 0$ for it to be a critical number. Now f has a local max at c means $f(x) \leq f(c)$ for all x near c which means that $f(x) - f(c) \leq 0$ near c. Because f is differentiable at c, $f'(c)$ exists, that is,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{exists.}$$

Consequently, both one-sided limits (derivatives) exist and are equal at c. Notice that the numerator $f(x) - f(c) \leq 0$ so

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \frac{- \text{ or } 0}{-} \geq 0$$

while

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \frac{- \text{ or } 0}{+} \leq 0.$$

Since both one-sided limits are equal, the only possibility is that they are both 0. In other words, $f'(c) = 0$ so c is a critical number.

□

YOU TRY IT 29.1. Do the proof for the case when f has a local min at c. What changes?

Caution: Notice that the CNT does NOT say that if c is a critical number, then f has a relative extrema at c. It is possible to have a zero derivative and not have an extreme point. It just says among all the critical points we will find the relative extrema.

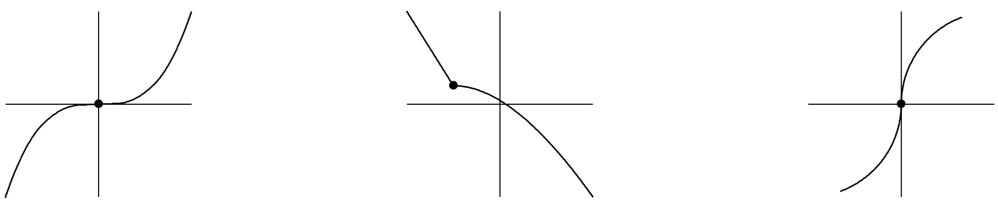


Figure 29.1: Three critical points which are not relative extreme values.

Putting Theorems Together. Now assume that f is continuous on the closed interval $[a, b]$. Then the EVT implies that f has an **absolute max** at some point c in $[a, b]$. There are two possibilities:

1. c is one of the endpoints, $c = a$ or $c = b$

2. OR c is between a and b... so $a < c < b$. But then c is not only an absolute max, it is also a **relative max** on (a, b). So the CNT implies that c is a critical number of f.

The same is true for an absolute min of f on $[a, b]$. Thus, we have proven

THEOREM 29.2 (CIT: The Closed Interval Theorem)**.** Let f be a continuous function on a closed interval $[a, b]$. Then the absolute extrema of f occur either at critical points of f on the open interval (a, b) or at the endpoints a and/or b.

From this we get:

Algorithm (Recipe) for Finding Absolute Extrema. To find the absolute extrema of a continuous function f on a closed interval $[a, b]$

1. Find the critical points of f on the open interval (a, b) and evaluate f at each such point.
2. Evaluate f at each of the endpoints, $x = a$ and $x = b$.
3. Compare the values; the largest is the absolute max and the smallest is the absolute min.

EXAMPLE 29.4. Find the absolute extrema of $p(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 1$ on $[0, 2]$.

SOLUTION. Notice that p is a polynomial so it is continuous and the interval is closed. So we can apply the CIT algorithm.

1. Find the critical points of p on the open interval $(0, 2)$ and evaluate p at each such point.
$$p'(x) = x^2 - 4x + 3 = (x - 3)(x - 1) = 0; \quad \text{at } x = 1, 3.$$
Notice that $x = 3$ lies outside the interval, so it is eliminated. $x = 1$ is the only critical point in $(0, 2)$. Finally, $p(1) = \frac{1}{3} - 2 + 3 + 1 = \frac{7}{3}$.
2. Evaluate f at each of the endpoints, $x = 0$ and $x = 2$: $p(0) = 1$ and $p(2) = \frac{8}{3} - 8 + 6 + 1 = \frac{5}{3}$.
3. The absolute max is $\frac{7}{3}$ at $x = 1$ and the absolute min is 1 at $x = 0$.

EXAMPLE 29.5. Find the absolute extrema of $f(x) = 3x^{2/3}$ on $[-1, 8]$.

SOLUTION. Notice that f is a root function so it is continuous and the interval is closed. So we can apply the CIT algorithm.

1. Find the critical points of f on $(-1, 8)$:
$$f'(x) = 2x^{-1/3} = \frac{2}{x^{1/3}} \neq 0.$$
However, $f'(x)$ DNE at $x = 0$. So $x = 0$ is a critical point in the interval. And $f(0) = 0$.
2. Evaluate f at the endpoints, $x = -1$ and $x = 8$: $f(-1) = -3$ and $f(8) = 4$.
3. The absolute max is 4 at $x = 2$ and the absolute min is 0 at $x = 0$.

EXAMPLE 29.6. Find the absolute extrema of $f(x) = x^2 e^{-x^2/4}$ on $[-1, 4]$.

SOLUTION. Notice that f is a the product of an exponential and a polynomial so it is continuous and the interval is closed. So we can apply the CIT algorithm.

1. Find the critical points of f on $(-1, 4)$:
$$f'(x) = 2xe^{-x^2/4} + x^2 e^{-x^2/4}(-\tfrac{2x}{4}) = 2xe^{-x^2/4} - \tfrac{1}{2}x^3 e^{-x^2/4} = e^{-x^2/4}(2x - \tfrac{1}{2}x^3) = 0.$$
So $2x - \frac{1}{2}x^3 = x(2 - \frac{1}{2}x^2) = 0 = 0$ at $x = 0$ or $2 = \frac{1}{2}x^2 \Rightarrow 4 = x^2$; $x = \pm 2$. So $x = 0$, 2 are the critical points in the interval. And $f(0) = 0$ while $f(2) = 4e^{-1} \approx 1.471$.
2. Evaluate f at the endpoints, $x = -1$ and $x = 4$: $f(-1) = e^{-1/4} \approx 0.7788$ and $f(4) = 16e^{-4} \approx 0.293$.
3. The absolute max is $4e^{-1}$ at $x = 2$ and the absolute min is 0 at $x = 0$.

YOU TRY IT 29.2. Find the absolute extrema of $f(x) = \ln(x^2 + 1)$ on $[-2, 1]$. [Answer: Global max of $\ln 5$ at $x = -2$ and global min of 0 at $x = 0$.]

YOU TRY IT 29.3. Find the absolute extrema of $f(x) = \sqrt{8x - x^2}$ on $[0, 6]$. [Answer: Global max of 4 at $x = 4$ and global min of 0 at $x = 0$.]

YOU TRY IT 29.4. Find the absolute extrema of $f(x) = xe^x$ on $[-2, 3]$. [Answer: Global max of $3e^3$ at $x = 3$ and global min of $-e{-}1$ at $x = -1$.]

EXAMPLE 29.7. Two non-negative numbers x and y sum to 10. What choices for x and y maximize e^{xy}?

SOLUTION. We want to maximize e^{xy}. The function has two variables. If we let $y = 10 - x$ then we want to find the absolute max of $f(x) = e^{x(10-x)} = e^{10x-x^2}$. Notice the possible values for x are 0 to 10, inclusive (since both x and y must be non-negative and since they sum to 10). So the interval is $[0, 10]$ and is closed. Since f is continuous we can apply the CIT algorithm.

1. Find the critical points of f on $(0, 10)$:

$$f'(x) = (10 - 2x)e^{10x-x^2} = 0 \text{ at } x = 5.$$

   So $x = 5$ is the only critical point in the interval. And $f(5) = e^{25}$.

2. Evaluate f at the endpoints, $x = 0$ and $x = 10$: $f(0) = e^0 = 1$ and $f(10) = e^0 = 1$.
3. The absolute max is e^{25} at $x = 5$.

30
The Mean Value Theorem

Introduction

Today we discuss one of the most important theorems in calculus—the MVT. It says something about the *slope* of a function on a closed interval based on the values of the function at the two endpoints of the interval. It relates *local* behavior of the function to its *global* behavior. This theorem turns out to be the key to many other theorems about the graphs of functions and their behavior. We begin with a simple case.

Rolle's Theorem

Rolle's theorem deals with functions that have the same starting and ending values.

THEOREM 30.1 (Rolle's Theorem). Assume that

1. f is continuous on the closed interval $[a, b]$;
2. f is differentiable on the open interval (a, b);
3. $f(a) = f(b)$, i.e., f has the same value at both endpoints.

Then there's some point c between a and b so that $f'(c) = 0$.

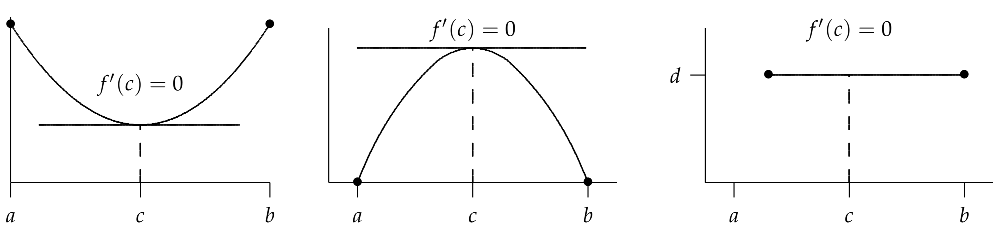


Figure 30.1: When the hypotheses of Rolle's theorem are satisfied, there is a horizontal tangent, i.e., a critical point exists.

EXAMPLE 30.1. How does Rolle's theorem apply to tossing a ball up and catching it at the same height? What about about walking a long a straight line and starting and ending at the same point?

Proof. By the CIT, f must have maximum and minimum values on $[a, b]$. There are two possibilities: either (i) both of these extreme values occur at the endpoints of $[a, b]$ or (ii) at least one of the extreme values occurs at a critical point of f.

In case (i), $f(a) = f(b) = d$ is both the minimum value and the maximum value of f on $[a, b]$. This can only happen if f is constant on the interval. But if f is constant, $f'(c) = 0$ for *any* point c between a and b. The situation in case (ii) is even easier. If f has an extreme point c between a and b, since f is differentiable, then by the CNT $f'(c) = 0$. $\square$

EXAMPLE 30.2. Show how Rolle's theorem applies to $f(x) = x^2 - 5x$ on $[1, 4]$.

SOLUTION. Check the three conditions

1. f is continuous on the closed interval $[1, 4]$ because it is a polynomial;
   f is differentiable on the open interval $(1, 4)$ again because it is a polynomial;
   $f(1) = -4$ and $f(4) = -4$, i.e., f has the same value at both endpoints.

So there is some point c between 1 and 4 so that $f'(c) = 0$. But $f'(x) = 2x - 5 = 0$ at $x = 2.5$. This, then is the value of c.

The Mean Value Theorem

Rolle's Theorem is used to prove the more general result, called the Mean Value theorem. **You should be able to state this theorem and draw a graph that illustrates it.**

THEOREM 30.2 (MVT: The Mean Value Theorem)**.** Assume that

1. f is continuous on the closed interval $[a, b]$;
2. f is differentiable on the open interval (a, b);

Then there is some point c in (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This is equivalent to saying $f(b) - f(a) = f'(c)(b - a)$.

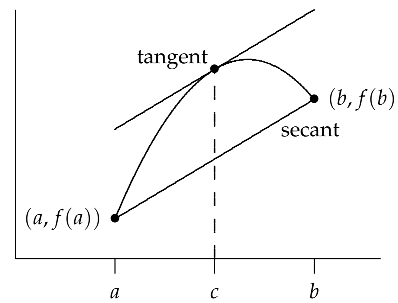


Figure 30.2: Under the conditions of the MVT, there's a point c in (a, b) so that the secant slope $f'(c) = \frac{f(b)-f(a)}{b-a}$ equals the tangent slope $f'(c)$.

Note: When $f(a) = f(b)$ we are back to Rolle's theorem... the conclusion of the MVT says in this case that $f'(c) = 0$.

Proof. Strategy: Modify f so that we can apply Rolle's theorem.

Let $\ell(x)$ be the (secant) line (to f) that passes through the points $(a, f(a))$ and $(b, f(b))$. Notice $\ell(x)$ is both continuous and differentiable everywhere (since it is a line). In fact, we know that $\ell'(x)$ is just the slope of the line which is (always)

$$\ell'(x) = \frac{f(b) - f(a)}{b - a}. \tag{30.1}$$

Now consider the difference function $g(x) = f(x) - \ell(x)$. Since f and ℓ are both continuous on $[a, b]$ so is g and since f and ℓ are both differentiable on (a, b) so is g.

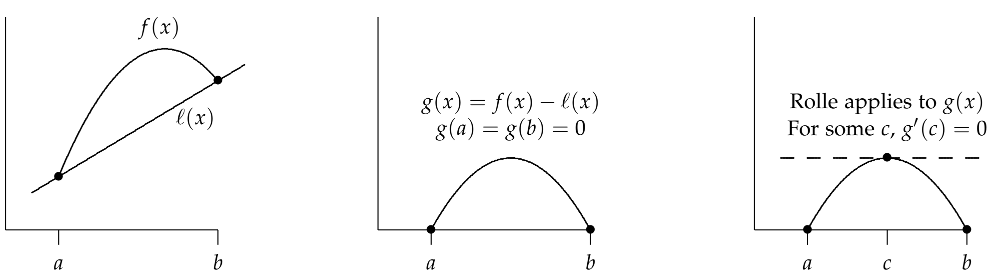


Figure 30.3: $f(x)$ and $\ell(x)$ have the same value at the endpoints a and b so their difference is 0 at both a and b. Consequently Rolle's theorem applies to g and there is a point c between a and b such that $g'(c) = 0$. .

Further

$$g(a) = f(a) - \ell(a) = f(a) - f(a) = 0 \text{ and } g(b) = f(b) - \ell(b) = f(b) - f(b) = 0.$$

So Rolle's theorem applies to g. This means there is a point c between a and b such that $g'(c) = 0$. But $g'(c) = f'(c) - \ell'(c) = 0$ which means using (30.1)

$$f'(c) = \ell'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

Mostly the MVT gets used to prove other theorems. But we can look at an example or two to see how it works.

EXAMPLE 30.3. Show how the MVT applies to $f(x) = x^3 - 6x + 1$ on $[0,3]$.

SOLUTION. Check the two conditions (hypotheses)

1. f is continuous on the closed interval $[0,3]$ because it is a polynomial;
   f is differentiable on the open interval $(0,3)$ again because it is a polynomial;

So the MVT applies: There is some point c between 0 and 3 so that

$$f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{10 - 1}{3} = 3.$$

Now

$$\begin{aligned} f'(x) = 3x^2 - 6 &= 3 \\ 3x^2 &= 9 \\ x &= \pm\sqrt{3} \end{aligned}$$

Only $x = \sqrt{3}$ is in the interval, so this is the value of c.

EXAMPLE 30.4. Show there does not exist a differentiable function on $[1,5]$ with $f(1) = -3$ and $f(5) = 9$ with $f'(x) \leq 2$ for all x.

SOLUTION. The MVT would apply to such a function f: So there is some point c between 1 and 5 so that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} = \frac{9 - (-3)}{4} = 3.$$

But supposedly $f'(x) \leq 2$ for all x. Contradiction. So no such f can exist.

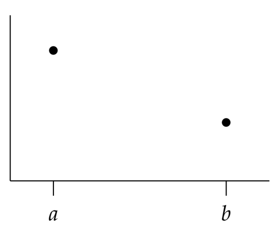
Using the MVT: Increasing and Decreasing Functions

As noted, the key value of the MVT is in proving other results.

THEOREM 30.3 (Increasing/Decreasing Test)**.** 1. If $f'(x) > 0$ for all x in an interval I, then f is increasing on I.

2. If $f'(x) < 0$ for all x in an interval I, then f is decreasing on I.

Proof. We'll prove (2). So assume $f'(x) < 0$ for all x in I. Let a and b be any two points in I with $a < b$. To show that f is decreasing, we need to show that $f(b) < f(a)$.



But f is differentiable on I so it is continuous on $[a,b]$ and differentiable on (a,b) so the MVT applies to f on $[a,b]$. So there's a point c in (a,b) so that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

or

$$f(b) - f(a) = f'(c)(b - a) = (-)(+) < 0.$$

But

$$f(b) - f(a) < 0 \Rightarrow f(b) < f(a)$$

which is what we wanted to show. $\square$

YOU TRY IT 30.1. Prove case (1) of the Increasing/Decreasing Test where $f'(x) > 0$.

EXAMPLE 30.5. Let $f(x) = x^4 - 6x^2 + 1$. Where is f increasing? Decreasing?

SOLUTION. Use the Increasing/Decreasing Test. Find the derivative and the critical points (where $f'(x) = 0$ or DNE).

$$f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 4x(x - \sqrt{3})(x + \sqrt{3}) = 0 \quad \text{at} \quad x = \pm\sqrt{3}, 0.$$

Now record this information on a number line for easy reference.

		0		0		0	
f'		$\vert$		$\vert$		$\vert$	
		$-(3^{1/2})$		0		$3^{1/2}$	

Now determine the sign of $f'(x)$ between and beyond the critical points. Here we use the IVT to know that the only places that the derivative can change sign are at the critical points because the derivative is continuous. Just plug in values in the appropriate intervals. $f'(-2) = -8$, $f'(-1) = +8$, $f'(1) = -8$, and $f'(2) = +8$.

	decreasing		increasing		decreasing		increasing
	$---$	0	$+++$	0	$---$	0	$+++$
f'		$\vert$		$\vert$		$\vert$	
		$-(3^{1/2})$		0		$3^{1/2}$	

Using interval notation: f is increasing on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ and it is decreasing on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.

EXAMPLE 30.6. Let $f(x) = xe^{2x}$. Where is f increasing? Decreasing?

SOLUTION. Use the Increasing/Decreasing Test. Find the derivative and the critical points.

$$f'(x) = e^{2x} + 2xe^{2x} = e^{2x}[1 + 2x] = 0 \quad \text{at} \quad x = -1/2.$$

Set up the number line and determine the sign of $f'(x)$ on either side of the critical point. $f'(-1) = -e^{-2} < 0$ and $f'(0) = 1$.

	decreasing		increasing
	$---$	0	$+++$
f'		$\vert$	
		$-1/2$	

Using interval notation: f is increasing on $(-1/2, \infty)$ and it is decreasing on $(-\infty, -1/2)$.

EXAMPLE 30.7. Let $f(x) = x - \sin x$. Where is f increasing? Decreasing?

SOLUTION. Use the Increasing/Decreasing Test. Find the derivative and the critical points.

$$f'(x) = 1 - \cos x = 0 \text{ at } x = 0, \pm 2\pi, \pm 4\pi \ldots.$$

Since $\cos x \leq 1$ the sign of $f'(x)$ between the critical points is always positive.

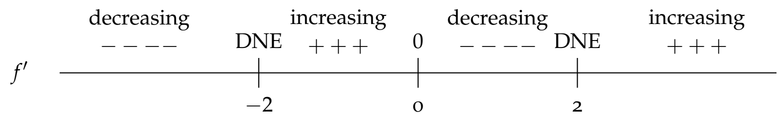
	increasing		increasing		increasing		increasing
	$+++$	0	$+++$	0	$+++$	0	$+++$
f'		$\vert$		$\vert$		$\vert$	
		-2π		0		2π	

EXAMPLE 30.8. Let $f(x) = (x^2 - 4)^{2/3}$. Where is f increasing? Decreasing?

SOLUTION. Use the Increasing/Decreasing Test. Find the derivative and the critical points.

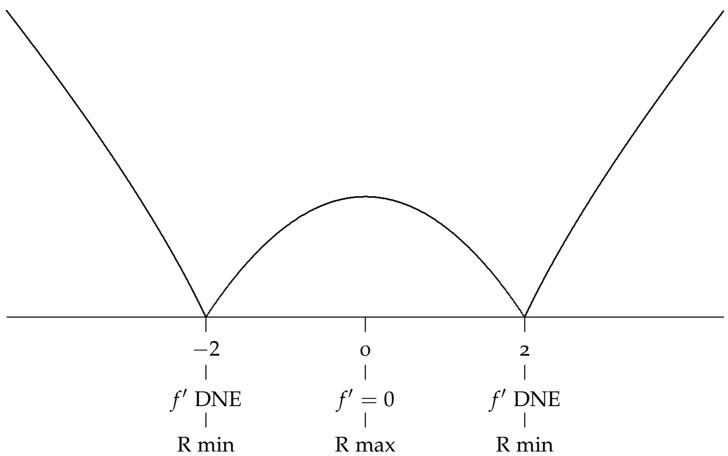
$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3} 2x = \frac{4x}{3(x^2 - 4)^{1/3}} = 0 \text{ at } x = 0 \text{ DNE at } x = \pm 2.$$

Determine the sign of $f'(x)$ between and beyond the critical points. $f'(-3) < 0$, $f'(-1) > 0$, $f'(1) < 0$ and $f'(3) > 0$.



Look at each of the critical points. Given how the function behaves on either side of each critical point, can you say which are relative extrema and what type? Justify your answer.

Can you sketch the shape of the graph based on the information?



31
The First Derivative Test

Because we now we know that

- $f'(x) > 0 \Rightarrow f$ is increasing
- $f'(x) < 0 \Rightarrow f$ is decreasing,

we can use this to classify a critical number as either local max or local min or neither as we did in the last example

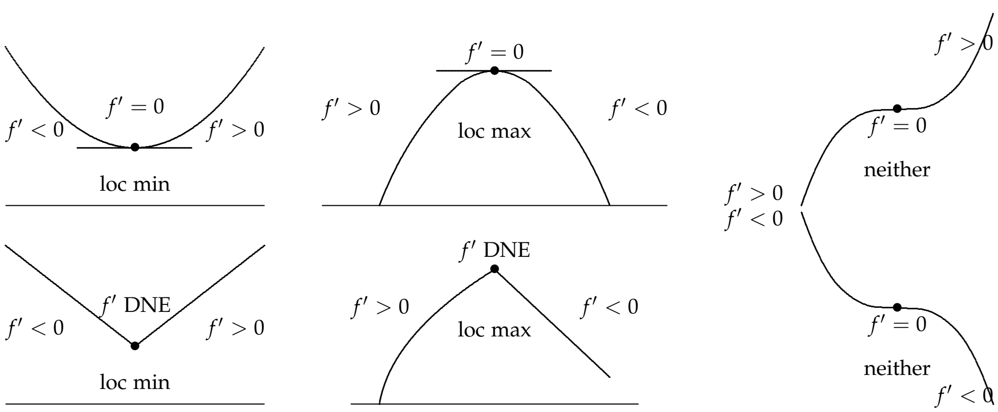


Figure 31.1: When the hypotheses of Rolle's theorem are satisfied, there is a horizontal tangent, i.e., a critical point exists.

Using the intuition from the Increasing/Decreasing Test, we obtain:

THEOREM 31.1 (The First Derivative Test)**.** Let c be a critical number of a continuous function f.

- If f' changes sign from positive to negative at c, then f has a local max at c.
- If f' changes sign from negative to positive at c, then f has a local min at c.
- If f' does not changes sign at c, then f has neither a local max nor min at c.

YOU TRY IT 31.1. Return to the example at the end of the last section and classify the critical points using the First Derivative Test.

EXAMPLE 31.1. Classify the relative extrema of $f(x) = x^4 - 6x^2 + 1$.

SOLUTION. We determined earlier the sign of $f'(x)$ along a number line. So now all we have to do is fill in the type of critical point.

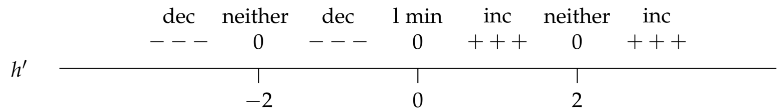
	dec	l min	inc	l min	dec	l max	inc
f'	$---$	0	$+++$	0	$---$	0	$+++$
		$-(3^{1/2})$		0		$3^{1/2}$	

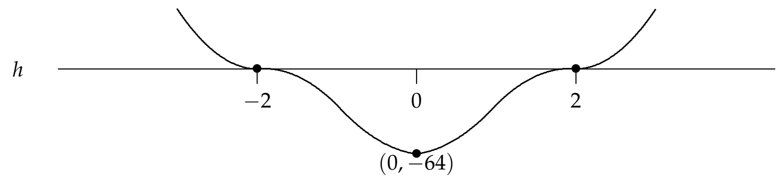
EXAMPLE 31.2. Classify the relative extrema of $h(x) = (x^2 - 4)^3$. Then sketch a quick graph plotting only the critical points.

SOLUTION. Use the First Derivative Test.

$$h'(x) = 3(x^2 - 4)^2(2x) = 0 \quad \text{at } x = 0, \pm 2.$$

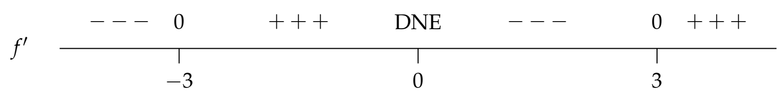
It is easy to determine the sign of the derivative.



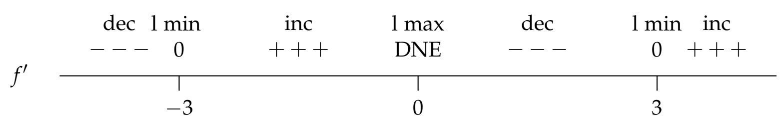


YOU TRY IT 31.2. Classify the relative extrema of $f(x) = \frac{1}{4}(x^2 - 9)^2$. [Answer: The critical values are at $x = 0, \pm 2$. Classify them.]

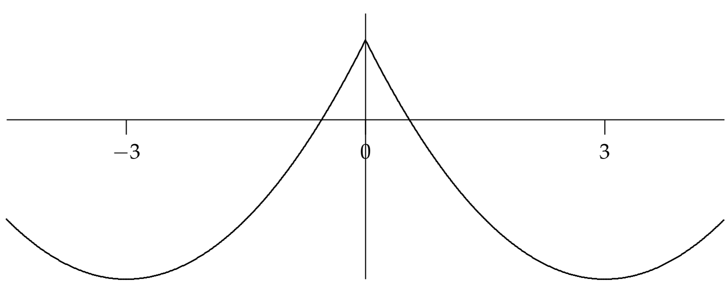
EXAMPLE 31.3. Suppose that a function $f(x)$ is continuous and has the following number line that describes its first derivative. Interpret this information to find where f is increasing, decreasing, and has relative extrema. Then draw a graph of the original function f that satisfies these conditions.



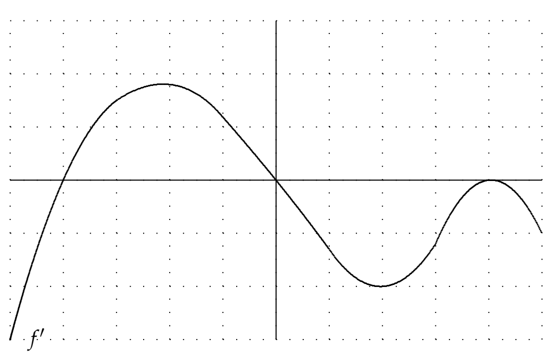
SOLUTION. Use the First Derivative Test to determine what type of extrema we have.



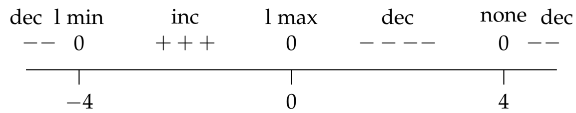
We can use this information to graph one possible solution for f. Note the function is cannot be differentiable at 0 (but should be elsewhere).



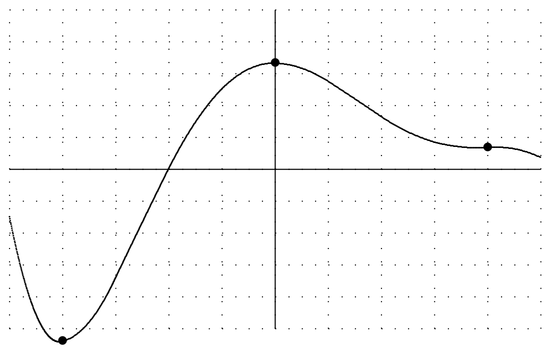
EXAMPLE 31.4. Suppose that the graph of f' is given below. Translate this information into ‘number line’ form and then attempt to graph the original function f.



SOLUTION. All we need to do is pay attention to the **sign** of the derivative.



Here's one function that satisfies these conditions. Notice that $x = 4$ is not a relative extreme point.



Working out the proof of the Mean Value Theorem

Rolle's Theorem is used to prove the more general result, called the Mean Value theorem. **You should be able to state this theorem and draw a graph that illustrates it.**

THEOREM 31.2 (MVT: The Mean Value Theorem)**.** Assume that

1. f is continuous on the closed interval $[a, b]$;
2. f is differentiable on the open interval (a, b);

Then there is some point c in (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This is equivalent to saying $f(b) - f(a) = f'(c)(b - a)$.

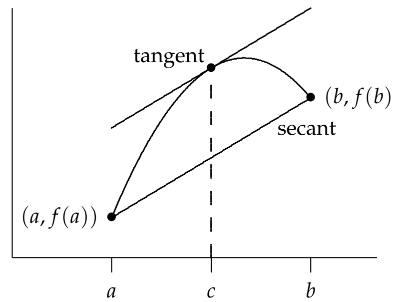


Figure 31.2: Under the conditions of the MVT, there's a point c in (a, b) so that the secant slope $f'(c) = \frac{f(b)-f(a)}{b-a}$ equals the tangent slope $f'(c)$.

Note: When $f(a) = f(b)$ we are back to Rolle's theorem... the conclusion of the MVT says in this case that $f'(c) = 0$.

Proof. Strategy: Modify f so that we can apply Rolle's theorem.

Let $\ell(x)$ be the (secant) line (to f) that passes through the points $(a, f(a))$ and $(b, f(b))$. Even though we have not figured out the equation for $\ell(x)$ we still know its derivative:

$$\ell'(x) = \text{slope of the line} = \underline{\hspace{8cm}}. \tag{31.1}$$

Since $\ell(x)$ is differentiable everywhere it is also ________________________.

Now consider the difference function $g(x) = f(x) - \ell(x)$. Since f and ℓ are both continuous on $[a, b]$, then g is also ________________________. Since f and ℓ are both differentiable on (a, b), then g is also ________________________.

Now check the values of g at the endpoints a and b:

$g(a) = f(a) - \ell(a) =$ ________________________ and $g(b) = f(b) - \ell(b) =$ ________________________.

So Rolle's theorem applies to g. This means there is a point c between a and b such that $g'(c) =$ __________. But then $g'(c) = f'(c) - \ell'(c) = 0$ which means using (31.1)

$$f'(c) = \ell'(c) = \underline{\hspace{6cm}}.$$

□

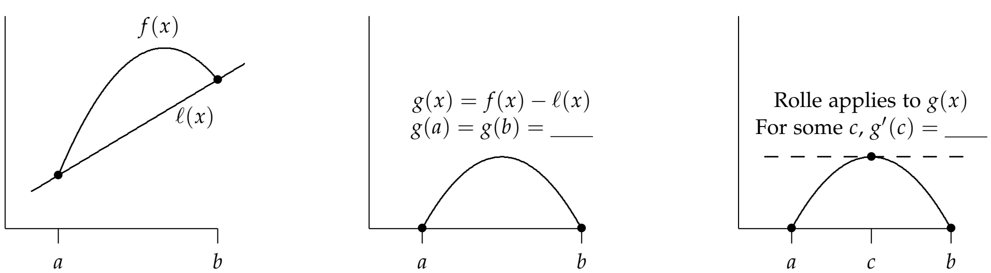


Figure 31.3: $f(x)$ and $\ell(x)$ have the same value at the endpoints a and b so their difference is ____ at both a and b. Consequently Rolle's theorem applies to g and there is a point c between a and b such that $g'(c) =$ ____.