Applications: Optimization

33.1 Functions With a Single Critical Number

Using the CIT we have been able to find the absolute extrema of the continuous functions on a *closed* interval. Using the First Derivative Test, we have found *relative* extrema for functions on their entire domains, often *open* intervals. However, in many of the applications that we are about to consider, we will be required to find the *absolute* extreme values of a function on an open interval. We can do this easily under the following circumstances.

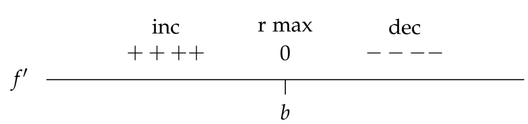
THEOREM 33.1 (SCPT: The Single Critical Point Theorem). Assume that f is differentiable[1] on an interval I (which may be open) and that b is the only critical number of f on I.

1. If f has a local max at b, then f actually has an absolute max at b.
2. If f has a local min at b, then f actually has an absolute min at b.

[1] Your text states a more general theorem where f only needs to be continuous. We assume more here and can actually use the MVT to give the proof.

Proof. We will do case (1). [Do case (2) as extra credit.] We are given that f has a local max at $x = b$. We want to show that if a is any other point in I, then $f(b) > f(a)$.

The first derivative test tells us that f' changes sign from positive to negative at b.



This means that

$$f'(x) > 0 \quad \text{when } x < b$$

and

$$f'(x) < 0 \quad \text{when } x > b.$$

So suppose that $a < b$. By the MVT there is some point c with $a < c < b$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or

$$f(b) - f(a) = f'(c)(b - a) = (\text{positive})(\text{positive}) > 0.$$

But

$$f(b) - f(a) > 0 \Rightarrow f(b) > f(a)$$

which is what we were trying to show. Instead, if $b < a$, then use the MVT again to get

$$f(b) - f(a) = f'(c)(b - a) = (\text{negative})(\text{negative}) > 0.$$

and again $f(b) > f(a)$. $\square$

EXAMPLE 33.1. Consider $y = f(x) = e^{-x^2}$ which is defined for all x or $(-\infty, \infty)$. It has a single critical number: $f'(x) = -2xe^{-x^2} = 0$ at $x = 0$.

	inc	l max	dec
	$++++$	0	$----$
f'		0	

By the SCPT, since $f(x) = e^{-x^2}$ has only one critical number and it is a relative max, then it has an absolute max at the point $x = 0$.

33.2 *Applied Optimization (Max-Min) Problems*

We now consider a wide variety of optimization problems of the sort we discussed briefly about a week ago.

EXAMPLE 33.2. A soup can is to hold 18 oz. What dimensions for the radius and height of the can *minimize* the cost (i.e., the materials) for the can. Here, **optimize** means minimize.

EXAMPLE 33.3. What tuition optimizes (maximizes) the revenue for HWS?

Overview

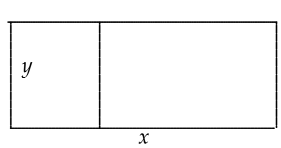
Such problems can be readily solved using the following analysis. Don't think of this as a recipe to memorize, rather try to understand the goal of each problem and then make use of the theory we have developed to solve the problem. Remember your solution is an essay or argument that in which you try to convince your reader (using evidence) that you have found a solution.

1. Identify the problem: Find the *absolute* max or min of a quantity Q [that is usually expressed as a function of two variables, which we call x and y].
2. Subject to a *constraint:* Usually a constant $k =$ a simple function of x and y.
3. Use the constraint to eliminate a variable, e.g., 'solve for y.'
4. Write Q as a function of one variable, $Q(x)$, and state its *domain*, which is often determined by the constraint equation. I cannot overemphasize determining the domain of $Q(x)$ in this process.
5. Find the absolute max or min of Q on its domain. The method you use will be determined, in part, by the domain. If the domain is a closed interval, you can use the CIT. If it is not closed, then you must hope that there is only one critical number so that you can use the SCPT. (The SCPT can also be used on a closed interval if there is a single critical number.) **Justify your answer in a sentence.**

   Notice that only in step (5) do we actually do any calculus.

33.3 *Lots of Examples*

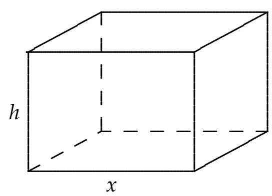
EXAMPLE 33.4. A real estate developer wishes to enclose a rectangular plot by a fence and then divide it into two plots by another fence parallel to one of the sides. What are the dimensions of the largest area that can be enclosed by using a total of 1800 meters of fencing?



SOLUTION. Let's use the guidelines to attack the problem.

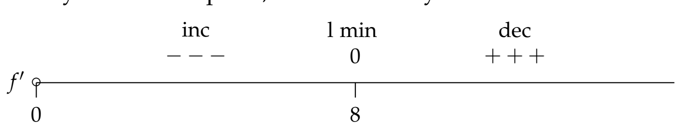
1. Maximize $A = xy$.
2. Subject to $P = 1800 = 2x + 3y$.
3. Eliminate x. $x = 900 - \frac{3}{2}y$. Notice that since the smallest x can be is 0, the largest y can be is 600 feet. And y is at least 0. Domain: $0 \le y \le 600$.
4. Write A in terms of a single variable with its domain:
$$A(y) = \left(900 - \frac{3}{2}y\right) y = 900y - \frac{3}{2}y^2 \qquad \text{on } [0, 600].$$
5. Solve the problem. Use CIT. $A' = 900 - 3y = 0$ at $y = 300$. Check the critical points and endpoints to determine the absolute max.
$A(300) = 300(450) = 135,000$ square feet.
$A(0) = 0$ and $A(600) = 0$.
By the CIT, A is maximized when $y = 600$ and $x = 450$. $A = 135,00$ square feet.

EXAMPLE 33.5. Find the dimensions of the box with a square base that has a volume of 512 in^3 and has the *smallest* surface area (4 sides, top and bottom).



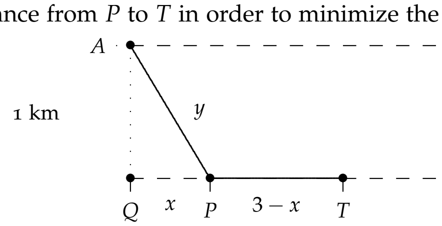
SOLUTION. Using the guidelines

1. Minimize surface $S = 4xh + 2x^2$
2. Subject to $V = 512 = x^2h$.
3. Eliminate h. $h = \frac{512}{x^2}$. Notice that x must be greater than 0 but can be as large as we like so the domain is $(0, \infty)$.
4. Rewrite S.
$$S = 4x\frac{512}{x^2} + 2x^2 = \frac{2048}{x} + 2x^2 \qquad \text{on } (0, \infty).$$
5. Since the interval is not closed, we must use the SCPT, if it has only one critical number.
$$S' = -\frac{2048}{x^2} + 4x = 0 \Rightarrow 4x = \frac{2048}{x^2} \Rightarrow x^3 = 512 \Rightarrow x = 8.$$
Yes, there is only one critical point, so now classify it.



Since there is only one critical point and it is a local min, by the SCPT it is also an absolute min. So the surface area is minimized when $x = 8$ and $h = \frac{512}{8^2} = 8$. The minimum surface area is $S = 4(8)(8) + 2(8^2) = 384$ square inches.

EXAMPLE 33.6. A cable television company has its master antenna located at point A on the bank of a straight river 1 kilometer wise (Figure 1 above). It is going to run a cable from A to a point P on the opposite bank of the river and then straight along the bank to a town T situated 3 kilometers downstream from A. It costs \$15 per meter to run the cable under the water and \$9 per meter to run the cable along the bank. What should be the distance from P to T in order to minimize the total cost of the cable?



SOLUTION. Using the guidelines

1. Minimize the cost $C = 15y + 9(3 - x)$
2. Subject to $y = \sqrt{x^2 + 1}$ where $0 \leq x \leq 3$.
3. Eliminate y. Here, there is nothing to do this time.
4. Rewrite C. $C(x) = 15\sqrt{x^2+1} + 9(3-x) \qquad \text{on } [0,3]$.
5. Use CIT (or possibly SCPT). Find the critical numbers.

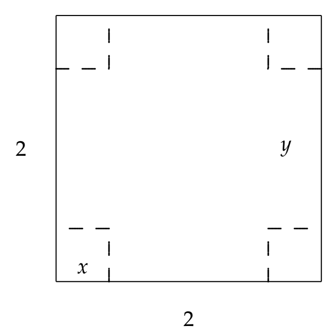
$$C'(x) = \frac{15x}{\sqrt{x^2+1}} - 9 = 0 \Rightarrow 15x = 9\sqrt{x^2+1} \Rightarrow 225x^2 = 81(x^2+1).$$

This means

$$144x^2 = 81 \Rightarrow 12x = \pm 9 \Rightarrow x = \pm 3/4.$$

Since $-3/4$ is not in the interval, so there is only a single critical point at $x = 3/4$. Use the CIT.
$C(3/4) = 15\sqrt{25/16} + 9(9/4) = 15(5/4) + 9(9/4) = 39$.
$C(0) = 15 + 27 = 42$.
$C(3) = 15\sqrt{10} \approx 47.4$.
So by the CIT, the absolute min occurs when $x = 3/4$ and the cost is 39.

EXAMPLE 33.7. A child's sandbox is to be made by cutting equal squares from the corners of a square sheet of aluminum and turning up the sides. If each side of the sheet of aluminum is 2 meters long, what size squares should be cut from the corners to maximize the volume of the sandbox?



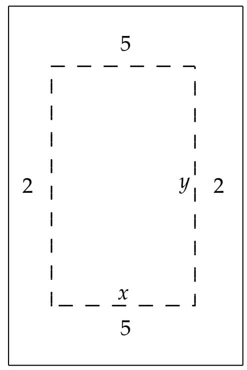
SOLUTION. This time

1. Maximize the volume $V = xy^2$
2. Subject to $2 = 2x + y$ where $0 \leq x \leq 1$.
3. Eliminate y. $y = 2 - 2x$.
4. Rewrite V. $V(x) = x(2-2x)^2 = 4x - 8x^2 + 4x^3 \qquad \text{on } [0,1]$.
5. Use CIT (or possibly SCPT). Find the critical numbers.

$$V'(x) = 4 - 16x + 12x^2 = 4(1 - 4x + 3x^2) = 4(1-x)(1-3x) = 0 \Rightarrow x = 1,\ 1/3.$$

Use the CIT.
$V(1/3) = \frac{1}{3}(\frac{4}{3})^2 = \frac{16}{27}$.
$V(0) = 0 = V(1)$.
So by the CIT, the absolute max occurs when $x = 1/3$ and the volume is $\frac{16}{27}$ cubic meters.

EXAMPLE 33.8. Advertising fliers are to be made from rectangular sheets of paper that contain 400 square centimeters of printed message. If the margins at the top and bottom are each 5 centimeters and the margins at the sides are each 2 centimeters, what should be the dimensions of the fliers if the total page area is to be a minimum?

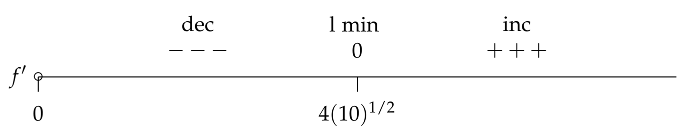


SOLUTION. This time

1. Minimize the total area $A = (x+4)(y+10)$
2. Subject to constraint $xy = 400$.
3. Eliminate y. $y = \frac{400}{x}$. x must be greater than 0 but may be as large as we like: domain $(0, \infty)$.
4. Rewrite A. $A(x) = (x+4)(\frac{400}{x} + 10) = 400 + 10x + \frac{1600}{x}$ on $(0, \infty)$.
5. Use SCPT. Find the critical numbers.

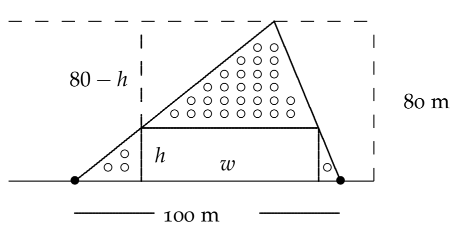
$$A'(x) = 10 - \frac{1600}{x^2} = 0 \Rightarrow x^2 = 160 \Rightarrow x = \pm 4\sqrt{10}.$$

Only $4\sqrt{10}$ is in the domain, so we can use SCPT. Classify the critical point using the First Derivative Test.



Since there is only one critical point and it is a local min, by the SCPT it is also an absolute min. So the area is minimized when $x = 4\sqrt{10}$. The flier dimensions should be $x + 4 = 4 + 4\sqrt{10} \approx 16.6$ in and $y + 10 = \frac{400}{4\sqrt{10}} = 10\sqrt{10} + 10 \approx 41.6$ in.

EXAMPLE 33.9. In New Geneva, two straight roads and a highway mark off a triangular piece of land (Figure 2 above). The side of the triangle lying along the highway is 100 meters long, and the perpendicular distance from the intersection of the two roads to the highway is 80 meters. Within this triangle, the city intends to zone a rectangular plot of land, fronting along the highway, for commercial use. The park department will plant shrubs in the remaining portion of the triangle. What is the maximum possible area of the rectangular plot of commercially zoned land?



SOLUTION. Using the outline

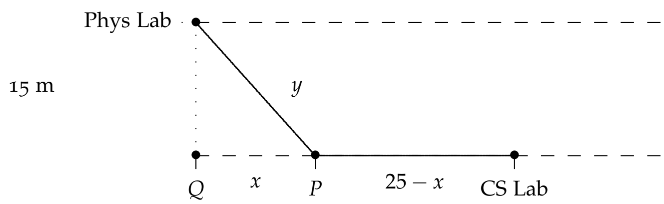
1. Maximize $A = hw$
2. Subject to the constraint (similar triangles): $\frac{w}{100} = \frac{80-h}{80} = 1 - \frac{1}{80}h$.
3. Eliminate w. $w = 100 - \frac{5}{4}h$. The smallest w can be is 0 and that makes $h = 80$. The largest w can be is 100 and that makes $h = 0$. So the domain for h is $[0, 80]$.
4. Rewrite $A = (100 - \frac{5}{4}h)h = 100h - \frac{5}{4}h^2$ on $[0, 80]$.

5. Use either CIT or SCPT if possible.

$$A' = 100 - \frac{5}{2}h = 0 \Rightarrow h = 40.$$

Use the CIT: Check the endpoints: $A(0) = 0$, $A(80) = 0$ and at the critical point $A(40) = (100 - \frac{5}{4}(40))(40) = 2000$ square ft. This the absolute max.

EXAMPLE 33.10 (A Key Max-Min Problem)**.** Optical fiber from a computer lab to a physics lab is being laid. The physics lab lies across a 15 m wide road and is 25 m down the other side from the computer lab. (See figure.) It costs \$13(00) per meter to put cable under the road and \$12(00) per meter to bury it underground. Where should the cable be brought across the road?



SOLUTION. Using the outline

1. Minimize the cost $C = 13y + 12(25 - x)$
2. Subject to $y^2 = x^2 + (15)^2$.
3. Eliminate y. $y = \sqrt{x^2 + 225}$ where $0 \leq x \leq 25$.
4. Rewrite C. $C(x) = 13\sqrt{x^2 + 225} + 12(25 - x) \qquad$ on $[0, 25]$.
5. Use CIT (or possibly SCPT). Find the critical numbers.

$$C'(x) = \frac{13x}{\sqrt{x^2 + 225}} - 12 = 0 \Rightarrow 13x = 12\sqrt{x^2 + 225} \Rightarrow 169x^2 = 144(x^2 + 225).$$

This means

$$25x^2 = (144)(225) \Rightarrow 5x = \pm 12(15) \Rightarrow x = \pm 36.$$

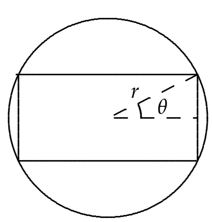
Since NEITHER point is in the interval. Use the CIT. The min occurs at an endpoint.

$C(0) = 13(15) + 12(25) = 495$.

$C(25) = 13\sqrt{625 + 225} = 379.01$.

So by the CIT, the absolute min occurs when $x = 0$ and the cost is \$379.01(00)=\$37,901.

EXAMPLE 33.11. Find the dimensions of the rectangle of largest perimeter that can be inscribed in a circle of radius r.



SOLUTION. Using the outline

1. Maximize the perimeter $P = 2x + 2y$
2. Subject to $\frac{y}{2} = r\sin\theta$ and $\frac{x}{2} = r\cos\theta$. Remember r is constant.
3. Eliminate both x and y!
4. Use Use $P = (4r\cos\theta)(4r\sin\theta) = 16r^2\cos\theta\sin\theta$ where θ is in the interval $[0, \pi/2]$.
5. Use CIT (or possibly SCPT).

$$P' = 16r^2(-\sin^2\theta + \cos^2\theta) = 0 \Rightarrow \cos^2\theta = \sin^2\theta \Rightarrow \cos\theta = \pm\sin\theta.$$

The only angle in $[0, \pi/2]$ where this is true is when $\theta = \pi/4$. Use CIT.

At the critical point: $P(\pi/4) = 16r^2 \cos(\pi/4)\sin(\pi/4) = 16r^2(\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) = 8r^2$.
At the endpoints: $P(0) = 16r^2\cos(0)\sin(0) = 16r^2(1)(0) = 0$
and $P(0) = 16r^2\cos(\pi/4)\sin(\pi/4) = 16r^2(0)(1) = 0$.
So the absolute max is $8r^2$ at $\theta = \pi/4$.

EXAMPLE 33.12. US Postal regulations `usps.com/consumers/domestic.htm` state: "Priority Mail is used for documents, gifts, and merchandise. The maximum size is 108 inches or less in combined length and distance around the thickest part (girth)." Find the dimensions of the box with square base of largest volume that can be sent Priority Mail.

SOLUTION. Using the outline

1. Maximize $V = x^2h$
2. Subject to $G = 108 = 4x + h$
3. Eliminate h. $h = 108 - 4x$ on $[0, 27]$.
4. Rewrite $V = x^2(108 - 4x) = 108x^2 - 4x^3$ on $[0, 27]$.
5. Use CIT (or perhaps SCPT). Find the critical numbers.

$$V' = 216x - 12x^2 = 12x(18 - x) = 0 \Rightarrow r = 0,\ 18.$$

Use CIT. Check V at the endpoints and critical numbers.
At the endpoints: $V(0) = 0$ and $V(27) = 0$.
At the critical number: $V(18) = 18^2(36) = 11{,}664$. So the absolute max occurs at $x = 18$ and $h = 108 - 72 = 36$.

EXAMPLE 33.13. US Postal regulations state: "Priority Mail is used for documents, gifts, and merchandise. The maximum size is 108 inches or less in combined length and distance around the thickest part (girth)." Find the dimensions of the cylinder of largest volume that can be sent Priority Mail.

SOLUTION. Using the outline

1. Maximize $V = \pi r^2 h$
2. Subject to $G = 108 = 2\pi r + h$
3. Eliminate h. $h = 108 - 2\pi r$ on $[0, \frac{54}{\pi}]$.
4. Rewrite $V = \pi r^2(108 - 2\pi r) = 108\pi r^2 - 2\pi^2 r^3$ on $[0, \frac{54}{\pi}]$.
5. Use CIT (or perhaps SCPT). Find the critical numbers.

$$V' = 216\pi r - 6\pi^2 r^2 = 6\pi r(36 - \pi r) = 0 \Rightarrow r = 0,\ \frac{36}{\pi}.$$

Use CIT. Check V at the endpoints and critical numbers.
At the endpoints: $V(0) = 0$ and $V(\frac{54}{\pi}) = 0$.
At the critical number: $V(\frac{36}{\pi}) = \pi(\frac{36}{\pi})^2(108 - 2\pi(\frac{36}{\pi})) > 0$. So the absolute max occurs at $r = \frac{36}{\pi}$ and $h = 108 - 2\pi(\frac{36}{\pi}) = 108 - 72 = 36$.

EXAMPLE 33.14. Find the dimensions of the cylinder of largest surface area that can be sent Priority Mail.

SOLUTION. This time maximize $A = 2\pi rh + 2\pi r^2$ (the tube plus the top and bottom).
As before, $h = 108 - 2\pi r$ on $[0, \frac{54}{\pi}]$.
So $A = 2\pi r(108 - 2\pi r) + 2\pi r^2 = 216\pi r - 4\pi^2 r^2$ on $[0, \frac{54}{\pi}]$. So

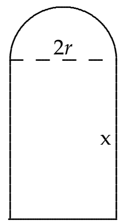
$$A' = 216\pi - 8\pi^2 r = 8\pi(27 - \pi r) = 0 \Rightarrow r = \frac{27}{\pi}.$$

Let's use SCPT. Justify: Use the first derivative test:

$$A' \quad \begin{array}{ccc} +++ & 0 & --- \\ \hline & \frac{27}{\pi} & \end{array}$$

there is a relative max at $r = \frac{27}{\pi}$. By the SCPT there is an absolute max there. $h = 108 - 2\pi(\frac{27}{\pi}) = 108 - 54 = 54$.

EXAMPLE 33.15. A *Norman window* is a window that consists of a rectangle surmounted[2] by a semicircle. These windows were commonly constructed in English architecture for about 100 years after the Norman conquest of Britain, circa 1066–1096.) If the perimeter is 24 ft, find the radius that will maximize the area of the window.



SOLUTION. Maximize: Area $A = 2rx + \frac{1}{2}\pi r^2$ (rectangle plus semicircle)

Subject to the constraint: $24 = 2x + 2r + \pi r$ (circumference)

Eliminate: $x = 12 - r - \frac{1}{2}\pi r = 12 - (1 + \frac{1}{2}\pi)r$ with domain: $0 \le r \le \frac{12}{1+(\pi/2)}$

Rewrite: $A = 2r(12 - [1 + \frac{1}{2}\pi]r) + \frac{1}{2}\pi r^2$ on $[0, \frac{12}{1+(\pi/2)}]$.

So $A = 24r - (2+\pi)r^2 + \frac{1}{2}\pi r^2 = 24r - (2 + \frac{1}{2}\pi)r^2$

Differentiate: $A' = 24 - (4+\pi)r = 0 \Rightarrow r = \frac{24}{4+\pi}$

Justify: Use the first derivative test:

$$A' \quad \begin{array}{ccc} +++ & 0 & --- \\ \hline & \frac{24}{4+\pi} & \end{array}$$

R. Max at $r = \frac{24}{4+\pi}$. Since there is only one critical number, there is an absolute max at $r = \frac{24}{4+\pi}$ by SCPT.

EXAMPLE 33.16. Soda cans hold 113π cm^3 (355 ml). The tops and bottoms are made of aluminum that is double thick. Find radius of the can that minimize the material used to make (sides, top, and bottom of) the can.

How does this compare to the radius of an actual soda can?

SOLUTION. This time minimize $A = 2\pi rh + 2(2\pi r^2)$ (the tube plus double top and bottom).

The constraint is $V = 113\pi = \pi r^2 h$.

Eliminate $h = \frac{113}{r^2}$ on $(0, \infty)$.

So $A = 2\pi r(\frac{113}{r^2}) + 4\pi r^2 = \frac{226\pi}{r} + 4\pi r^2$ on $(0, \infty)$. We expect to use SCPT.

$$A' = -\frac{226\pi}{r^2} + 8\pi r = 0 \Rightarrow 8\pi r = \frac{226\pi}{r^2} \Rightarrow r^3 = \frac{113}{4} \Rightarrow r = \sqrt[3]{\frac{113}{4}}.$$

Use SCPT. First classify the critical number. Use the first derivative test:

$$A' \quad \begin{array}{ccc} --- & 0 & +++ \\ \hline & \left(\frac{113}{4}\right)^{1/3} & \end{array}$$

There is a relative min at $r = \sqrt[3]{\frac{113}{4}}$. By the SCPT there is an absolute min there.

Measuring a real can we see that the radius is approximately 3.15 cm. Comparing $\sqrt[3]{\frac{113}{4}} \approx \sqrt[3]{\frac{113}{4}} \approx 3.05$.

[2] This may be one of the few remaining uses in English of the word "surmount" in reference to physical objects.