Graphing with Asymptotes

35.1 Vertical Asymptotes

Over the next few classes we will return to graphing functions. This time, besides the usual critical points, local extrema, and inflections, we will also consider functions that have vertical asymptotes (VA) that we discussed earlier in the term and functions that have horizontal asymptotes. Recall our earlier definition:

DEFINITION 35.1. The function $f(x)$ has a **vertical asymptote** (VA) at $x = a$ if

$$\lim_{x\to a^+} f(x) = +\infty \text{ or } -\infty \quad \text{and/or} \quad \lim_{x\to a^-} f(x) = +\infty \text{ or } -\infty.$$

EXAMPLE 35.1. Find the VA's of $f(x) = \frac{x^2}{x-1}$.

SOLUTION. With rational functions we look to see where the denominator is 0 and then take the appropriate (one-sided) limits. Here, we look at $x \to 1$.

$$\lim_{x\to 1^+} \frac{x^2}{x-1} \to \frac{1}{0^+} = +\infty.$$

So we know that $f(x)$ has a VA at $x = 1$. Checking the other limit,

$$\lim_{x\to 1^-} \frac{x^2}{x-1} \to \frac{1}{0^-} = -\infty.$$

EXAMPLE 35.2. Find the VA's of $g(x) = \frac{x^2+3x-10}{x-2}$.

SOLUTION. This time, we look as $x \to 2$.

$$\lim_{x\to 2+} \frac{x^2+3x-10}{x-2} = \lim_{x\to 2+} \frac{(x+5)(x-2)}{x-2} = \lim_{x\to 2+} (x+5) = 7.$$

In fact, now that we have factored we see that the two-sided limit exists.

$$\lim_{x\to 2} \frac{x^2+3x-10}{x-2} = \lim_{x\to 2} \frac{(x+5)(x-2)}{x-2} = \lim_{x\to 2+} (x+5) = 7.$$

So $g(x)$ does not have a VA. Rather it has a removable discontinuity at $x = 2$.

DEFINITION 35.2. The function $f(x)$ has a **removable discontinuity** at $x = a$ if $\lim_{x\to a} f(x)$ exists (and is finite) but does not equal $f(a)$.

EXAMPLE 35.3. Do a complete graph (including VA's) of $f(x) = \frac{x^2}{x-1}$. $(x \neq 1)$

SOLUTION. As usual, start with critical points.

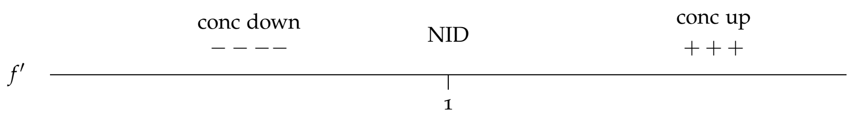
$$f'(x) = \frac{2x(x-1)-x^2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2} = 0 \text{ at } x = 0, 2; \text{ NID } x = 1$$

where NID means the point is not in the domain.

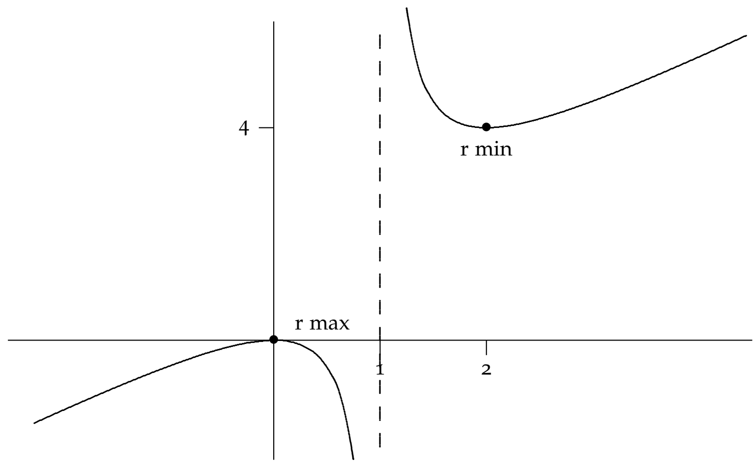
		0		1		2	
f'	inc $+++$	l max 0	dec $--$	NID	dec $--$	l min 0	inc $+++$

Check concavity with the second derivative. After simplifying:

$$f''(x) = \frac{2}{(x-1)^3} \neq 0; \text{ NID } x = 1.$$

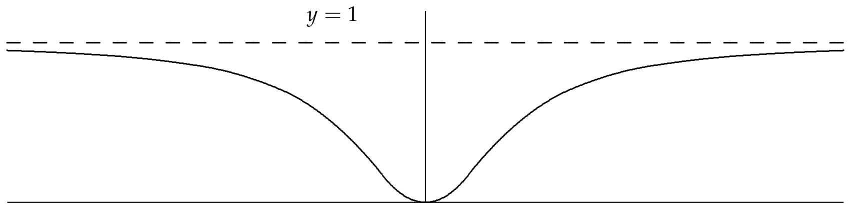


We have already found that there is a VA at $x = 1$. Plot the critical numbers; there are no inflections: $f(0) = 0$ and $f(2) = 4$. Make sure to mark the VA in the graph.



35.2 *Horizontal Asymptotes*

Now consider the function $f(x) = \frac{x^2}{x^2+1}$ which is graphed below. It does not have any vertical asymptotes but it does have a horizontal asymptote at $y = 1$.



So what do we mean when we say that f has a horizontal asymptote at $y = 1$? Something like:

"As $x \to \infty$, $f(x)$ gets close to $y = 1$"
"As $x \to -\infty$, $f(x)$ gets close to $y = 1$"

We can see this by looking at a table of values:

x	± 1	± 10	± 100	$\pm 1,000$
$\frac{x^2}{x^2+1}$	$\frac{1}{2}$	$\frac{100}{101}$	$\frac{10,000}{10,001}$	$\frac{1,000,000}{1,000,001}$

The following informal definition will be sufficient for our purposes.

DEFINITION 35.3 (Limits at Infinity). We say that

$$\lim_{x \to +\infty} f(x) = L$$

if we can make $f(x)$ arbitrarily close to L by taking x sufficiently large. Similarly,

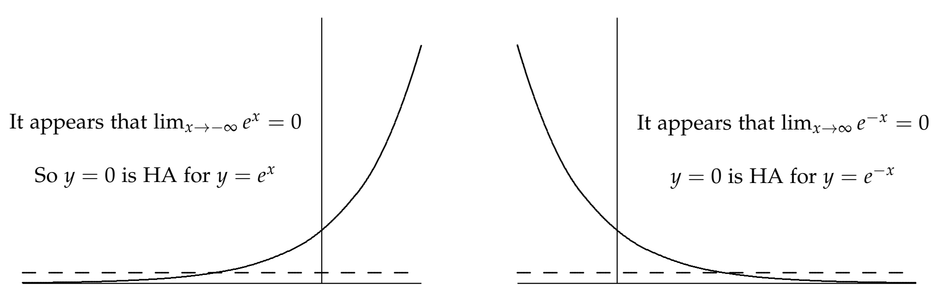
$$\lim_{x \to -\infty} f(x) = M$$

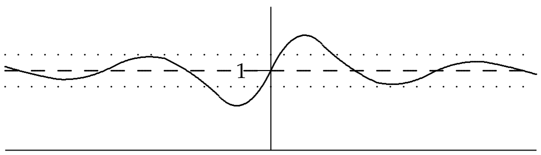
if we can make $f(x)$ arbitrarily close to M by taking x sufficiently large in magnitude but negative.

DEFINITION 35.4. The line $y = L$ is a **horizontal asymptote** (HA) for the graph of $f(x)$ if either $\lim_{x \to +\infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

Pictures

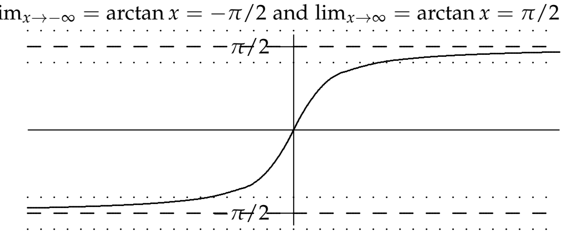
Here are a few graphs of functions with horizontal asymptotes. Notice that the function can cross a horizontal asymptote (but not a vertical one). Notice that if $y = L$ is a horizontal asymptote, then it means when x is large enough (in one direction or the either) the function stays within a little horizontal corridor about the line $y = L$... i.e., $f(x)$ gets close to L.





$\lim_{x\to\infty} f(x) = 1$ so $y = 1$ is an HA for $y = f(x)$

$\lim_{x\to-\infty} = \arctan x = -\pi/2$ and $\lim_{x\to\infty} = \arctan x = \pi/2$



$y = \pm\pi/2$ are HA's for $y = \arctan x$

Working with Limits at Infinity

Here's a simple function $f(x) = \frac{1}{x}$. As x gets large in magnitude it is easy to see that $f(x)$ approaches 0.

x	± 10	± 100	$\pm 1,000$	$\pm 1,000,000$
$\frac{1}{x}$	$\pm .1$	$\pm .01$	$\pm .001$	$\pm .000001$

In other words we have the following

FACT 35.1. Let $f(x) = \frac{1}{x}$, then

$$\lim_{x\to\infty} \frac{1}{x} = 0 \text{ and } \lim_{x\to-\infty} \frac{1}{x} = 0.$$

This means that $y = 0$ is an HA for $f(x) = \frac{1}{x}$.

Most of the basic limit laws carry over for limits at infinity. So we can use them to show: if $r > 0$ (even fractional values of r are fine), then

$$\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to\infty} \left(\frac{1}{x}\right)^r = \left(\lim_{x\to\infty} \frac{1}{x}\right)^r = (0)^r = 0.$$

So, if c is any constant, then

$$\lim_{x\to\infty} \frac{c}{x^r} = c \cdot \lim_{x\to\infty} \frac{1}{x^r} = c \cdot 0 = 0.$$

FACT 35.2. If $r > 0$, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0$$

and, as long as x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

This fact makes sense; just think about it. If x gets large, then x^r gets large (if $r > 0$) so $\frac{c}{x^r}$ gets small.

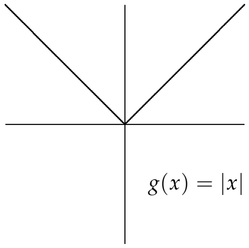
There are two other limits that you ought to memorize, though you should already know them from the graphs of these functions.

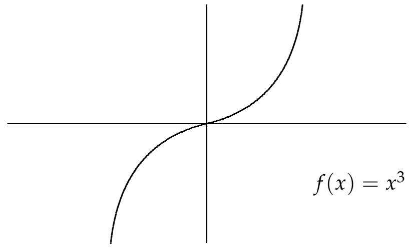
FACT 35.3.

$$\lim_{x\to-\infty} e^x = 0 \text{ and } \lim_{x\to\infty} e^{-x} = 0.$$

35.3 *Infinite Limits at Infinity*

Many simple and familiar functions get very large in magnitude as x itself gets large in magnitude We say that such functions have an **infinite limit at infinity**. A couple of familiar examples include $f(x) = x^3$ and $g(x) = |x|$ as illustrated below.





DEFINITION 35.5 (Infinite Limits at Infinity). If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

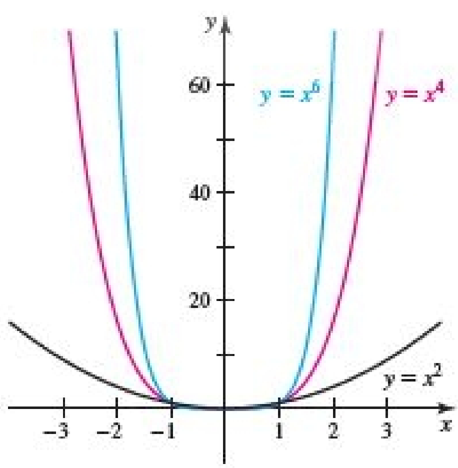
If $f(x)$ becomes arbitrarily large in magnitude but negative as x becomes arbitrarily large, then we write

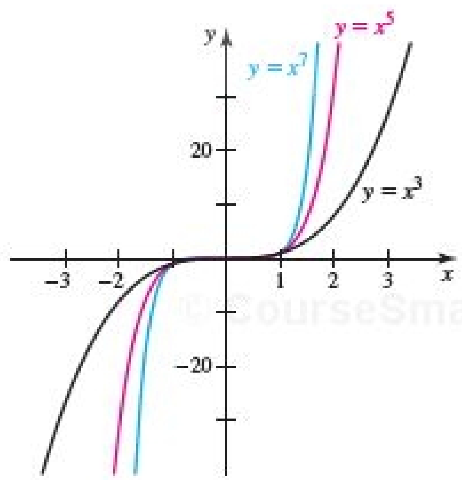
$$\lim_{x\to\infty} f(x) = -\infty.$$

Similar definitions are used for $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

The End Behavior of Polynomials

Infinite limits at infinity describe the behavior of all polynomials of degree greater than 0. The simplest examples are provided by functions of the form $f(x) = x^n$ where n is a positive integer. Since positive powers of large numbers are large, this means that for all n, $\lim_{x\to\infty} x^n = +\infty$. Limits at $-\infty$ are only slightly more complicated. Since we are now looking at powers of large magnitude *negative* numbers, the product will be either positive or negative depending on the number of terms, but the magnitude will always be large. More specifically, when n is *even* x^n is always non-negative, so $\lim_{x\to-\infty} x^n = +\infty$. Since a product of an odd number of negative numbers is odd, $\lim_{x\to-\infty} x^n = -\infty$. This behavior is illustrated below for a few odd and even powers of x.





Using these observations, it is not hard to show that when we have any polynomial, its behavior as $x \to \pm\infty$ is completely determined by its highest power. That is if $p(x)$ is a degree n polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$ and $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n$. It is worth gathering all of these observations together in a theorem, though they should seem quite intuitive or natural.

THEOREM 35.1 (Limits of Powers and Polynomials). Let n be a positive integer. Then

(1) If n is even, then $\lim_{x\to\infty} x^n = +\infty$ and $\lim_{x\to-\infty} x^n = +\infty$.

(2) If n is odd, then $\lim_{x\to\infty} x^n = +\infty$ and $\lim_{x\to-\infty} x^n = -\infty$.

(3) If $p(x)$ is a degree n polynomial, then $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$ and $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n$, where $a_n x^n$ is the highest degree term.

Note: A polynomial does not have any horizontal asymptotes.

EXAMPLE 35.4 (Polynomial Limits at Infinity). A couple of simple examples illustrate the ideas in the theorem. Let $p(x) = 9x^4 - 2x + 1$ and $q(x) = -4x^5 + 7x^2 + 3$. Then

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} 9x^4 - 2x + 1 = \lim_{x\to\infty} 9x^4 = +\infty$$

because the degree is 4 (even) and the leading coefficient is 9 (positive). Similarly

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} 9x^4 - 2x + 1 \lim_{x\to-\infty} 9x^4 = +\infty$$

because the degree is still even and the leading coefficient is 9 (positive). Now

$$\lim_{x\to\infty} q(x) = \lim_{x\to\infty} -4x^5 + 7x^2 + 3 = \lim_{x\to\infty} -4x^5 = -\infty$$

because the degree is 3 (odd) and the leading coefficient is -3 (negative). Similarly

$$\lim_{x\to-\infty} q(x) = \lim_{x\to-\infty} -4x^5 + 7x^2 + 3 = \lim_{x\to-\infty} -4x^5 = +\infty$$

because the degree is 3 (odd) and the leading coefficient is -3 (negative) and the limit is approaching negative infinity.

These are relatively easy... just think about the sign of the highest degree term as $x \to +\infty$ or $x \to -\infty$.

35.4 *HA's and Rational Functions*

The key to finding horizontal asymptotes for rational functions is to *divide the numerator and denominator by x to the degree (highest power of x) of the denominator*.

EXAMPLE 35.5. Find the HA's of $f(x) = \frac{4x^4-2x^3+7}{3x^4+2x^2+1}$.

SOLUTION. We need to determine the limits as $x \to \pm\infty$. Dividing by x^4 (the degree of the denominator is 4)

$$\lim_{x\to\infty} \frac{4x^4 - 2x^3 + 7}{3x^4 + 2x^2 + 1} = \lim_{x\to\infty} \frac{4 - \frac{2}{x} + \frac{7}{x^4}}{3 + \frac{2}{x^2} + \frac{1}{x^4}} = \frac{4 - 0 + 0}{3 + 0 + 0} = \frac{4}{3}.$$

So $y = \frac{4}{3}$ is an HA. What about as $x \to -\infty$?

$$\lim_{x\to-\infty} \frac{4x^4 - 2x^3 + 7}{3x^4 + 2x^2 + 1} = \lim_{x\to-\infty} \frac{4 - \frac{2}{x} + \frac{7}{x^4}}{3 + \frac{2}{x^2} + \frac{1}{x^4}} = \frac{4}{3}.$$

EXAMPLE 35.6. Find $\lim_{x\to-\infty} \frac{3x^2-2x}{9x^3+1}$.

SOLUTION. Dividing by x^3 (the degree of the denominator is 3)

$$\lim_{x\to-\infty} \frac{3x^2 - 2x}{9x^3 + 1} = \lim_{x\to-\infty} \frac{\frac{3}{x} - \frac{2}{x^2}}{9 + \frac{1}{x^3}} = \frac{0 - 0}{9 + 0} = 0.$$

So $y = 0$ is an HA.

EXAMPLE 35.7. Find $\lim_{x\to-\infty} \frac{4x^3-2}{2x^2+x}$.

SOLUTION. Dividing by x^2 (the degree of the denominator is 2)

$$\lim_{x\to-\infty}\frac{4x^3-2}{2x^2+x}=\lim_{x\to-\infty}\frac{4x-\frac{2}{x^2}}{2+\frac{1}{x}}=\lim_{x\to-\infty}\frac{4x}{2}=-\infty.$$

There is no HA.

EXAMPLE 35.8. Find $\lim_{x\to\infty}\frac{2\sqrt{x}}{2x+1}$.

SOLUTION. Dividing by x (the degree of the denominator is 1)

$$\lim_{x\to\infty}\frac{2x^{1/2}}{2x+1}=\lim_{x\to\infty}\frac{\frac{2}{x^{1/2}}}{2+\frac{1}{x}}=\frac{0}{2+0}=0.$$

There is an HA at $y=0$.

Dominant Powers Here are a couple of quick observations we can make about a rational function $\frac{p(x)}{q(x)}$ from these examples. The limits depend on the highest degree terms in both the numerator and denominator. So we can focus on just those terms. For example:

$$\lim_{x\to-\infty}\frac{3x^2-2x+1}{2x^4+6x}=\lim_{x\to-\infty}\frac{3x^2}{2x^4}=\lim_{x\to-\infty}\frac{\frac{3}{x^2}}{2}=\frac{0}{2}=0.$$

Similarly

$$\lim_{x\to\infty}\frac{2x^3+2x^2+1}{5x^3+x}=\lim_{x\to\infty}\frac{2x^3}{5x^3}=\frac{2}{5}.$$

Or

$$\lim_{x\to\infty}\frac{2x^{3/2}+x-1}{5x+7}=\lim_{x\to\infty}\frac{2x^{3/2}}{5x}=\lim_{x\to\infty}\frac{2x^{1/2}}{5}=+\infty.$$

The general pattern is clear:

THEOREM 35.2 (Limits at Infinity for Rational Functions)**.** Let $p(x)$ and $q(x)$ be polynomials.

(1) If the degree of the numerator is *less* than the degree of the denominator, then $\lim_{x\to\pm\infty}\frac{p(x)}{q(x)}=0$ and $x=0$ is a HA.

(2) If the degree of the numerator is *the same* as the degree of the denominator, then $\lim_{x\to\pm\infty}\frac{p(x)}{q(x)}=\frac{a}{b}$ where a and b are the leading coefficients of p and q and $x=\frac{a}{b}$ is a HA.

(3) If the degree of the numerator is *larger* than the degree of the denominator, then $\lim_{x\to\pm\infty}\frac{p(x)}{q(x)}=\infty$ or $-\infty$ depending on the highest powers and their coefficients in the polynomials $p(x)$ and $q(x)$. There is no HA.

EXAMPLE 35.9. Now we will use these observations to evaluate the following rational functions

$$\lim_{t\to+\infty}\frac{3t-1}{t^3-1}=0$$

because the degree of the numerator is 1 and the denominator is 3. Next

$$\lim_{x\to+\infty}\frac{2x^3+x^2+1}{4x^3-2}=\frac{2}{4}=\frac{1}{2}$$

because the degrees of the numerator and denominator are equal.

$$\lim_{x\to-\infty}\frac{5x^5-x}{10x^2+1}=\lim_{x\to-\infty}\frac{5x^5}{10x^2}=\lim_{x\to-\infty}2x^3=-\infty.$$

Here the degree of the numerator is larger than the degree of the denominator.