Graphing Rational Functions

Let's use all of the material we have developed to graph some rational functions.

EXAMPLE 37.11. Graph $y = f(x) = \frac{x^2+3x-3}{x^2}$. Include both vertical and horizontal asymptotes.

SOLUTION. First determine the domain: $f(x)$ is rational and is not defined where the denominator is 0. That's at $x = 0$. This leads us to look VAs, RDs, and HAs
VA: SInce the function is not defined at $x = 0$, we look to see if there is a VA there.

$\lim_{x\to 0^+} \frac{\overbrace{x^2+3x-3}^{-3}}{\underbrace{x^2}_{0^+}} = -\infty$ and $\lim_{x\to 0^-} \frac{\overbrace{x^2+3x-3}^{-3}}{\underbrace{x^2}_{0^+}} = -\infty$; so VA at $x = 0$. You may wish to indicate the VA in your graph at this point.

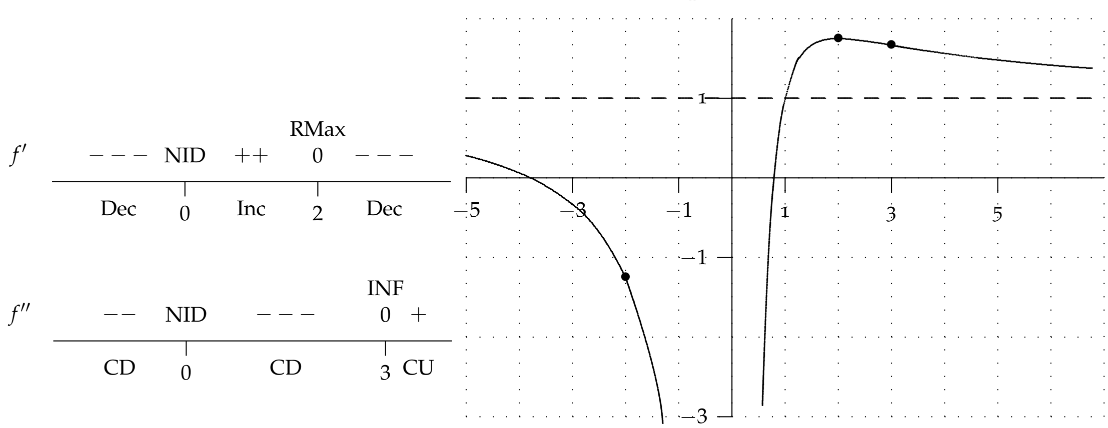
HA and End Behavior: Using dominant powers, $\lim_{x\to+\infty} \frac{x^2+3x-3}{x^2} = \lim_{x\to+\infty} \frac{x^2}{x^2} = 1$ and $\lim_{x\to-\infty} \frac{x^2+3x-3}{x^2} = \lim_{x\to-\infty} \frac{x^2}{x^2} = 1$. So HA at $y = 1$. You may wish to indicate the HA in your graph at this point.

Critical points, local extrema, increasing/decreasing behavior.
$f'(x) = \frac{(2x+3)x^2 - (x^2+3x-3)2x}{x^4} = \frac{(2x^2+3x-3) - 2(x^2+3x-3)}{x^3} = \frac{-3x+6}{x^3} =$ 0 at $x = 2$ (and $x = 0$ NID).

Inflections and concavity.
$f''(x) = \frac{-3x^3 - (-3x+6)3x^2}{x^6} = \frac{-3x - (-3x+6)(3)}{x^4} = \frac{6x-18}{x^4} = 0$ at $x = 3$ (and $x = 0$ NID).

Evaluate f at key points. $f(2) = \frac{4+6-3}{4} = \frac{7}{4}$, $f(3) = \frac{9+9-3}{9} = \frac{5}{3}$.
Notice that the inflection is almost imperceptible in the graph.

We will need another point to graph when $x < 0$. $f(-2) = \frac{4-6-3}{4} = -1.25$.



EXAMPLE 37.12. Graph $y = f(x) = \frac{x^2}{x-4}$. Include both vertical and horizontal asymptotes, if they exist.

SOLUTION. Notice that $x = 4$ is not in the domain. Check there for a VA.

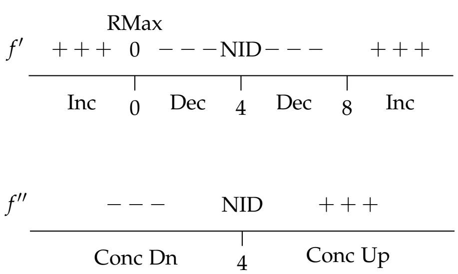
VA: $\lim_{x\to 4+} \frac{\overbrace{x^2}^{16}}{\underbrace{x-4}_{0^+}} = +\infty$ and $\lim_{x\to 4-} \frac{\overbrace{x^2}^{16}}{\underbrace{x-4}_{0^-}} = -\infty$; so VA at $x = 4$.

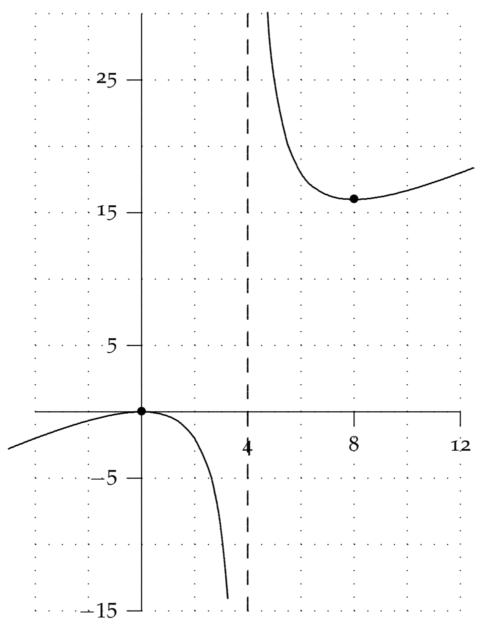
HA and End Behavior: Since the function is rational, using dominant powers, $\lim_{x\to+\infty} \frac{x^2}{x-4} = \lim_{x\to+\infty} \frac{x}{1-4/x} = +\infty$ and $\lim_{x\to-\infty} \frac{x^2}{x-4} = \lim_{x\to-\infty} \frac{x}{1-4/x} = -\infty$. So there are no HAs, but we do know what is happening at either end.

Critical points, local extrema, increasing/decreasing behavior.
$f'(x) = \frac{2x(x-4)-x^2}{(x-4)^2} = \frac{x^2-8x}{(x-4)^2} = \frac{x(x-8)}{(x-4)^2} = 0$ at $x = 0, 8$ (and $x = 4$ NID).

Inflections and concavity.
$f''(x) = \frac{(2x-8)(x-4)^2-(x^2-8x)(2)(x-4)}{(x-4)^4} = \frac{(2x-8)(x-4)-(x^2-8x)(2)}{(x-4)^3} = \frac{(2x^2-16x+32-2x^2+16x)}{(x-4)^3} = \frac{32}{(x-4)^3} \neq 0$ (but $x = 4$ NID). So there are no inflections. But the concavity may still switch on either side of $x = 4$.

Evaluate f at key points. $f(0) = 0$ and $f(8) = \frac{64}{8-4} = 16$.





EXAMPLE 37.13. Graph $y = f(x) = \frac{2x}{x+2}$. Include both vertical and horizontal asymptotes.

SOLUTION. This time $x = -2$ is not in the domain.

VA: $\lim_{x\to -2^+} \frac{\overbrace{2x}^{-4}}{\underbrace{x+2}_{0^+}} = -\infty$ and $\lim_{x\to -2^+} \frac{\overbrace{2x}^{-4}}{\underbrace{x+2}_{0^-}} = +\infty$; so VA at $x = -2$.

HA: Using dominant powers, $\lim_{x\to+\infty} \frac{2x}{x+2} = \lim_{x\to+\infty} \frac{2x}{x} = 2$ and $\lim_{x\to-\infty} \frac{2x}{x+2} = \lim_{x\to-\infty} \frac{2x}{x} = 2$. So HA at $y = 2$.

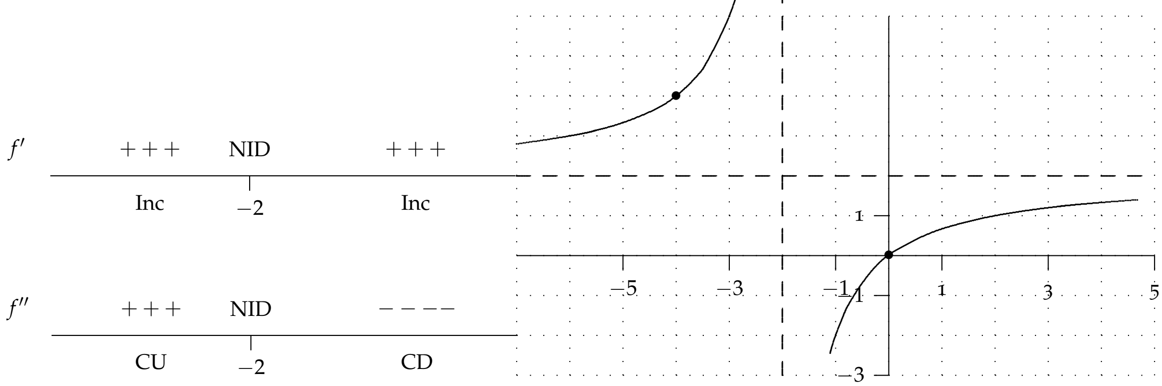
Critical points, local extrema, increasing/decreasing behavior.

$f'(x) = \frac{2(x+2) - 2x}{(x+2)^2} = \frac{4}{(x+2)^2} \neq 0$ (and $x = -2$ NID).

Inflections and concavity.

$f''(x) = \frac{-8}{(x+2)^3} \neq 0$ (and $x = -2$ NID).

Evaluate f at key points. There are none! Choose on each side of VA. $f(0) = 0$ and $f(-4) = \frac{-8}{-2} = 4$.



EXAMPLE 37.14. Graph $y = f(x) = \frac{2x}{x^2+1}$. Include both vertical and horizontal asymptotes.

SOLUTION. VA: None, the denominator of this rational function is never 0.

HA: Using dominant powers, $\lim_{x\to+\infty} \frac{2x}{x^2+1} = \lim_{x\to+\infty} \frac{2x}{x^2} = \lim_{x\to+\infty} \frac{\frac{2}{x}}{1} = 0$ and $\lim_{x\to-\infty} \frac{2x}{x^2+1} = \lim_{x\to-\infty} \frac{2x}{x^2} = \lim_{x\to-\infty} \frac{\frac{2}{x}}{1} = 0$. So HA at $y = 0$.

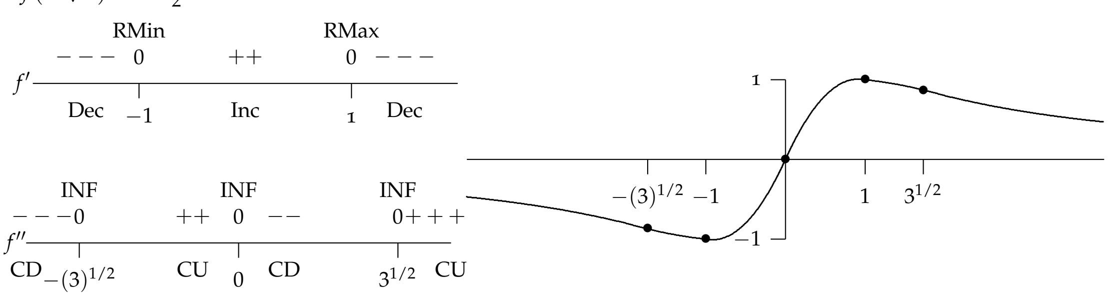
Critical points, local extrema, increasing/decreasing behavior.

$f'(x) = \frac{2(x^2+1) - (2x)2x}{(x^2+1)^2} = \frac{2 - 2x^2}{(x^2+1)^2} = 0$ at $x = \pm 1$.

Inflections and concavity.

$f''(x) = \frac{-4x(x^2+1)^2 - (2-2x^2)2(x^2+1)2x}{(x^2+1)^4} = \frac{4x^3 - 12x}{(x^2+1)^3} = \frac{4x(x^2-3)}{(x^2+1)^3} = 0$ at $x = 0, \pm\sqrt{3}$.

Evaluate f at key points. $f(1) = 1$, $f(-1) = -1$, $f(0) = 0$, $f(\sqrt{3}) = \frac{\sqrt{3}}{2} \approx 0.866$ and $f(-\sqrt{3}) = -\frac{\sqrt{3}}{2} \approx -0.866$.



EXAMPLE 37.15. Graph $y = f(x) = \frac{2x^2}{(x-1)^2}$, where $x \neq 1$. Include both vertical and horizontal asymptotes.

SOLUTION. The function is not defined at $x = 1$.

VA: Look near $x = 1$. $\lim_{x\to1^+} \frac{\overbrace{2x^2}^{1}}{\underbrace{(x-1)^2}_{0^+}} = +\infty$ and $\lim_{x\to1^-} \frac{\overbrace{2x^2}^{1}}{\underbrace{(x-1)^2}_{0^+}} = +\infty$.

HA: Using dominant powers,

$$\lim_{x\to+\infty} \frac{2x^2}{(x-1)^2} = \lim_{x\to+\infty} \frac{2x^2}{x^2-2x+1} = \lim_{x\to+\infty} \frac{2x^2}{x^2} = 2$$

and

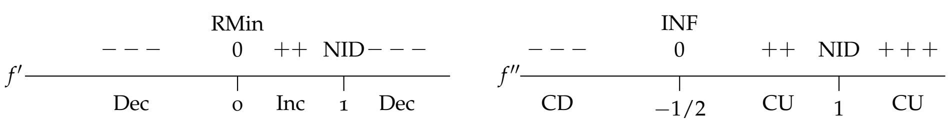
$$\lim_{x\to-\infty} \frac{2x^2}{(x-1)^2} = \lim_{x\to-\infty} \frac{2x^2}{x^2} = 2.$$

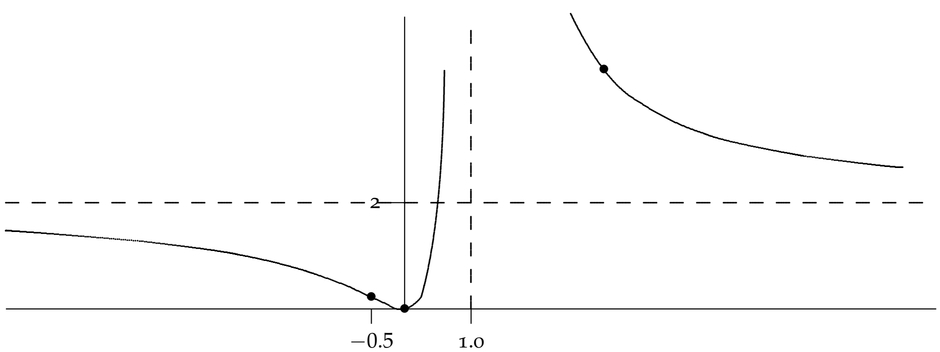
So HA at $y = 2$.

Critical points, local extrema, increasing/decreasing behavior.
$f'(x) = \frac{4x(x-1)^2 - (2x^2)2(x-1)}{(x-1)^4} = \frac{4x^2-4x-4x^2}{(x-1)^3} = \frac{-4x}{(x-1)^3} = 0$ at $x = 0$, $x = 1$ NID.

Inflections and concavity.
$f''(x) = \frac{-4(x-1)^3 - 4x(3)2(x-1)^2}{(x-1)^6} = \frac{-4x+4+12x}{(x-1)^4} = \frac{8x+4}{(x-1)^4} = 0$ at $x = -1/2$, $x = 1$ NID.

Evaluate f at key points. $f(0) = 0$, $f(-1/2) = 2/9$ and we need a point when $x > 1$ on the other side of the VA: $f(3) = 18/4 = 4.5$.





EXAMPLE 37.16. Graph $y = f(x) = \frac{x}{x^2-4}$, where $x \neq \pm 2$. Include both vertical and horizontal asymptotes.

SOLUTION. The function is not defined at $x = \pm 2$.

VA: Look near $x = 2$. $\lim_{x\to 2^+} \frac{\overbrace{x}^{2}}{\underbrace{x^2-4}_{0^+}} = +\infty$ and $\lim_{x\to 2^+} \frac{\overbrace{x}^{2}}{\underbrace{x^2-4}_{0^-}} = -\infty$. Now look near $x = -2$. $\lim_{x\to -2^+} \frac{\overbrace{x}^{-2}}{\underbrace{x^2-4}_{0^-}} = +\infty$ and $\lim_{x\to -2^-} \frac{\overbrace{x}^{-2}}{\underbrace{x^2-4}_{0^+}} = -\infty$ VAs: $x = \pm 2$

HA: Using dominant powers,

$$\lim_{x\to+\infty} \frac{x}{x^2-4} = \lim_{x\to+\infty} \frac{x}{x^2} = \lim_{x\to+\infty} \frac{1}{x} = 0$$

and

$$\lim_{x\to-\infty} \frac{x}{x^2-4} = \lim_{x\to-\infty} \frac{1}{x} = 0$$

So HA at $y = 0$.

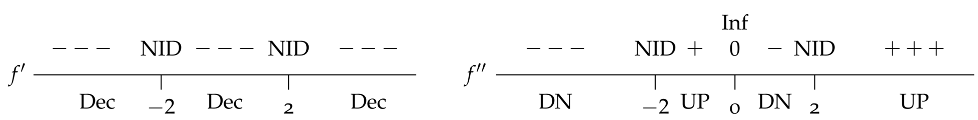
Critical points, local extrema, increasing/decreasing behavior.

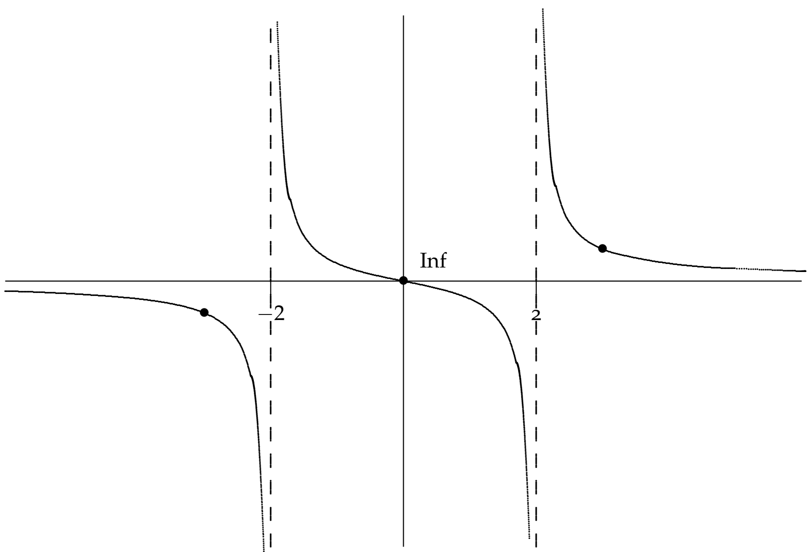
$f'(x) = \dfrac{x^2 - 4 - 2x^2}{(x^2-4)^2} = \dfrac{-x^2-4}{(x^2-4)^2} \neq 0$, $x = \pm 2$ NID.

Inflections and concavity.

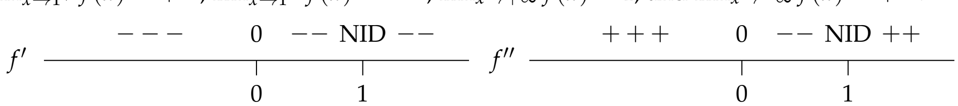
$f''(x) = \dfrac{-2x(x^2-4)^2 + (x^2+4)2(x^2-4)(2x)}{(x^2-4)^4} = \dfrac{-2x^3+8x+4x^3+16x}{(x^2-4)^3} =$
$\dfrac{2x^3+24x}{(x^2-4)^3} = \dfrac{2x(x^2+12)}{(x^2-4)^3} = 0$ at $x = 0$, $x = \pm 2$ NID.

Evaluate f at key points. $f(0) = 0$, and we need a points when $x > 2$ and $x < -2$ on the far side of the VAs :$f(3) = 3/5$ and $f(-3) = -3/5$.





YOU TRY IT 37.1. Here is information about the first and second derivatives of a function and its vertical and horizontal asymptotes. Sketch a function that satisfies these conditions. Indicate on your graph which points are local extrema and which are inflections. **NID** means the point is "not in the domain" of the original function. Let $f(0) = -1$ and $\lim_{x\to1^+} f(x) = +\infty$, $\lim_{x\to1^-} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = 1$, and $\lim_{x\to-\infty} f(x) = +\infty$.



38
Indeterminate Forms and l'Hôpital's Rule

Most of the interesting limits we have seen so far have had the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and we have had to do "more work" to evaluate them. This work might have been factoring, using conjugates, using known limits or dividing by the highest power of x. We will now introduce another method of "work" that helps us deal with these limits.

Remember that we say that such limits have **indeterminate form**. We start with three types:

1. $\dfrac{0}{0}$: $\displaystyle\lim_{x\to 2}\frac{x^2-4}{x-2}$
2. $\dfrac{\infty}{\infty}$: $\displaystyle\lim_{x\to\infty}\frac{2x^2-4}{3x^2+9}$
3. and a new type $0\cdot\infty$: $\displaystyle\lim_{x\to\infty} xe^{-x}$

There are other types, as well. The new method is called

THEOREM 38.1 (l'Hôpital's Rule)**.** Let f and g be differentiable on an open interval I containing c (except perhaps at c itself). Assume that $g'(c)\neq 0$ (except perhaps at c). IF

(a) both $\displaystyle\lim_{x\to c} f(x)=0$ and $\displaystyle\lim_{x\to c} g(x)=0$ OR both $\displaystyle\lim_{x\to c} f(x)=\pm\infty$ and $\displaystyle\lim_{x\to c} g(x)=\pm\infty$
AND
(b) both $\displaystyle\lim_{x\to c}\frac{f'(x)}{g'(x)}$
THEN

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

This also applies to one-sided limits and to limits as $x\to\infty$ or $x\to-\infty$

EXAMPLE 38.1. We could evaluate the following indeterminate limit the old way:

$$\lim_{x\to 2}\frac{x^2-4^{\nearrow 0}}{x-2_{\searrow 0}}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2} x+2=4.$$

But we could also use l'Hôpital's rule:

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{2x}{1}=4$$

which is pretty easy. Similarly for an indeterminate form of $\frac{\infty}{\infty}$, consider

$$\lim_{x\to\infty}\frac{2x^2-4^{\nearrow\infty}}{3x^2+9_{\searrow\infty}}=\lim_{x\to\infty}\frac{4x}{6x}=\frac{2}{3}.$$

This technique can be applied to problems where our old techniques failed. Try these

1. $\lim_{x\to 1} \dfrac{1-x^{\nearrow 0}}{\ln x_{\searrow 0}} = \lim_{x\to 1} \dfrac{-1}{\frac{1}{x}} = \lim_{x\to 1} -x = -1$

2. $\lim_{x\to 0} \dfrac{1-\cos 3x^{\nearrow 0}}{2x^2_{\searrow 0}} = \lim_{x\to 0} \dfrac{3\sin 3x^{\nearrow 0}}{4x_{\searrow 0}} = \lim_{x\to 0} \dfrac{9\cos 3x}{4} = \dfrac{9}{4}$

3. $\lim_{x\to 0} \dfrac{x^2+x^{\nearrow 0}}{e^x_{\searrow 1}} = \lim_{x\to 0} \dfrac{2x+1}{e^x} = \dfrac{2}{1} = 2$

4. $\lim_{x\to\infty} \dfrac{3x^2+7x^{\nearrow\infty}}{5x^2+11_{\searrow\infty}}$

5. $\lim_{x\to\infty} \dfrac{-x^{2\nearrow -\infty}}{e^x_{\searrow\infty}}$

6. $\lim_{x\to\infty} \dfrac{\ln x^{\nearrow\infty}}{e^x_{\searrow\infty}}$

7. $\lim_{x\to\infty} \dfrac{\ln x^{\nearrow\infty}}{x_{\searrow\infty}} \lim_{x\to\infty} \dfrac{\frac{1}{x}}{1} = 0$

8. $\lim_{x\to 0^+} \dfrac{\ln x^{\nearrow -\infty}}{\frac{1}{x}_{\searrow +\infty}} = \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \dfrac{x}{-1} = 0.$

The Indeterminate Form: $0\cdot\infty$. Now here's an application to the a new type of indeterminate form: The limit $\lim_{x\to 0^+} x\ln x$ has form $0\cdot\infty$. Rewriting it we can apply l'Hôpital's rule.

9. $\lim_{x\to 0^+} x\ln x = \dfrac{\ln x^{\nearrow\infty}}{\frac{1}{x}_{\searrow\infty}}$ which we just did in #8

Try these

10. $\lim_{x\to\infty} x^2 e^{-x} =$

11. $\lim_{x\to\infty} x\sin(\frac{1}{x}) \to \infty\cdot 0$. But

$$\lim_{x\to\infty} x\sin(\tfrac{1}{x}) = \lim_{x\to\infty} \frac{\sin(\frac{1}{x})^{\nearrow 0}}{\frac{1}{x}_{\searrow 0}} = \lim_{x\to\infty} \frac{-\frac{1}{x^2}\cos(\frac{1}{x})}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\cos(\frac{1}{x})}{1} = \cos 0 = 1.$$

12. $\lim_{x\to\infty} x\tan(\frac{1}{x}) =$

EXAMPLE 38.2. Graph $y = f(x) = \frac{2x+e^x}{e^x}$. Include both vertical and horizontal asymptotes.

SOLUTION. HA: Use l'Hôpital's rule:

$$\lim_{x\to+\infty} \frac{2x+e^x}{e^x} = \lim_{x\to+\infty} \frac{2+e^x}{e^x} = \lim_{x\to+\infty} \frac{e^x}{e^x} = 1$$

So HA at $y = 1$. Also

$$\lim_{x\to-\infty} \frac{(2x+e^x)^{\nearrow -\infty}}{e^x_{\searrow 0^+}} = -\infty$$

$$f'(x) = \frac{(2+e^x)e^x - (2x+x)e^x}{(e^x)^2} = \frac{(2+e^x)-(2x+e^x)}{e^x} = \frac{2-2x}{e^x} = 0 \text{ at } x = 1.$$

$$f''(x) = \frac{-2e^x - (2-2x)e^x}{(e^x)^2} = \frac{-2-(2-2x)}{e^x} = \frac{-4+2x}{e^x} = 0 \text{ at } x = 2.$$

Evaluate f at key points. $f(1) = \frac{2+e}{e} \approx 1.736$ and $f(2) = \frac{4+e^2}{e^2} \approx 1.541$.