

Antiderivatives

40.1 Introduction

So far much of the term has been spent finding derivatives or rates of change. But in some circumstances we already know the rate of change and we wish to determine the original function. For example, meters or data loggers often measure rates of change, e.g., miles per hour or kilowatts per hour.

If you know the velocity of an object, can you determine the position of the object. This could happen in a car, say, where the speedometer readings were being recorded. Can the position of the car be determined from this information? Similarly, can the position of an airplane be determined from the black box which records the airspeed?

More generally, given $f'(x)$ can we find the function $f(x)$. If you think about it, this is the sort of question I have asked you to do on labs, tests, and homework assignments where I gave you the graph of $f'(x)$ and said draw the graph of $f(x)$. Or where I gave you the number line information for $f'(x)$ and $f''(x)$ and asked you to reconstruct the graph of $f(x)$. Can we do this same thing if we start with a formula for $f'(x)$? Can we get an explicit formula for $f(x)$? We usually state the problem this way.

DEFINITION 40.1. Let $f(x)$ be a function defined on an interval I . We say that $F(x)$ is an **antiderivative** of $f(x)$ on I if

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

EXAMPLE 40.1. If $f(x) = 2x$, then $F(x) = x^2$ is an antiderivative of f because

$$F'(x) = 2x = f(x).$$

But so is $G(x) = x^2 + 1$ or, more generally, $H(x) = x^2 + c$.

Are there ‘other’ antiderivatives of $f(x) = 2x$ besides those of the form $H(x) = x^2 + c$? We can use the MVT to show that the answer is ‘No.’ The proof will require three small steps.

THEOREM 40.1 (Theorem 1). If $F'(x) = 0$ for all x in an interval I , then $F(x) = k$ is a constant function.

This makes a lot of sense: If the velocity of an object is 0, then its position is constant (not changing). Here’s the

Proof. To show that $F(x)$ is constant, we must show that any two output values of F are the same, i.e., $F(a) = F(b)$ for all a and b in I .

So pick any a and b in I (with $a < b$). Then since F is differentiable on I , then F is both continuous and differentiable on the smaller interval $[a, b]$. So the MVT

applies. There is a point c between a and b so that

$$\frac{F(b) - F(a)}{b - a} = F'(c) \Rightarrow F(b) - F(a) = F'(c)[b - a] = 0[b - a] = 0.$$

This means

$$F(b) = F(a),$$

in other words, F is constant. □

THEOREM 40.2 (Theorem 2). Suppose that F, G are differentiable on the interval I and $F'(x) = G'(x)$ for all x in I . Then there exist k so that

$$G(x) = F(x) + k.$$

Proof. Consider the function $G(x) - F(x)$ on I . Then

$$\frac{d}{dx} (G(x) - F(x)) = G'(x) - F'(x) = 0.$$

Therefore, by Theorem 1

$$G(x) - F(x) = k$$

so

$$G(x) = F(x) + k.$$

□

THEOREM 40.3 (Theorem 3: Families of Antiderivatives). If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval I , then $G(x) = F(x) + k$.

This is the theorem we want to show.

Proof. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval I then $G'(x) = f(x)$ and $F'(x) = f(x)$, that is, $F'(x) = G'(x)$ on I . Then by Theorem 2 $G(x) = F(x) + k$. □

DEFINITION 40.2. If $F(x)$ is any antiderivative of $f(x)$, we say that $F(x) + c$ is the **general antiderivative** of $f(x)$ on I .

Notation for Antiderivatives

Antidifferentiation is also called ‘indefinite integration.’

$$\int f(x) dx = F(x) + c.$$

- \int is the integration symbol
- $f(x)$ is called the integrand
- dx indicates the variable of integration
- $F(x)$ is a particular antiderivative of $f(x)$
- and c is the constant of integration.
- We refer to $\int f(x) dx$ as an ‘antiderivative of $f(x)$ ’ or an ‘indefinite integral of f .’

Here are several examples.

$$\int 2x \, dx = x^2 + c$$

$$\int \cos t \, dt = \sin t + c$$

$$\int e^z \, dz = e^z + c$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + c$$

Antidifferentiation reverses differentiation so

$$\int F'(x) \, dx = F(x) + c$$

and differentiation undoes antidifferentiation

$$\frac{d}{dx} \left[\int f(x) \, dx \right] = f(x).$$

Differentiation and antidifferentiation are reverse processes, so each derivative rule has a corresponding antidifferentiation rule.

Differentiation

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(kx) = k$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Antidifferentiation

$$\int 0 \, dx = c$$

$$\int k \, dx = kx + c$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$$

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \sec x \tan x \, dx = \sec x + c$$

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + c$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + c$$

Variations and Generalizations

Notice what happens when we use ax instead of x in some of these functions. We multiply by a when taking the derivative, so we have to divide by a when taking the antiderivative.

Differentiation

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}(\sin ax) = a \cos ax$$

$$\frac{d}{dx}(\tan ax) = a \sec^2 ax$$

$$\frac{d}{dx}(\arcsin(\frac{x}{a})) = \frac{1}{\sqrt{a^2-x^2}}$$

$$\frac{d}{dx}(\arctan(\frac{x}{a})) = \frac{a}{a^2+x^2}$$

Antidifferentiation

$$\int e^{ax} \, dx = \frac{1}{a}e^{ax} + c$$

$$\int \cos ax \, dx = \frac{1}{a} \sin x + c$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \arcsin(\frac{x}{a}) + c$$

$$\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \arctan(\frac{x}{a}) + c$$

Try filling in the rules for $\int \sin ax \, dx$ and $\int \sec(ax) \tan(ax) \, dx$.

EXAMPLE 40.2. Here are a few examples.

$$\begin{aligned}\int \cos(4x) dx &= \frac{1}{4} \sin(4x) + c \\ \int e^{z/2} dz &= 2e^{z/2} + c \\ \int \frac{1}{16+x^2} dx &= \frac{1}{4} \arctan \frac{x}{4} + c\end{aligned}$$

40.2 Problems

1. Determine antiderivatives of the following functions. Take the derivative of your answer to confirm that you are correct. Why should you add $+c$ to any answer? **Basics:**

(a) $7x^6$	(b) x^6	(c) $2x^6$	(d) e^x	(e) $2e^x$
(f) $2e^{2x}$	(g) e^{2x}	(h) $\frac{1}{x}$	(i) $-\frac{8}{x}$	
(j) $\cos x$	(k) $4 \cos x$	(l) $4 \cos(4x)$	(m) $\cos(4x)$	(n) $-\frac{1}{x} + 4 \cos x$
(o) $-2 \sec x \tan x$	(p) $\sin 4x$	(q) $4 \sec^2 x$	(r) $2x$	(s) x
(t) $8x$	(u) $6^x \ln 6$	(v) $6x - 6^x \ln 6$		
(w) 6	(x) $6 + \sec^2 x$	(y) $\frac{6}{1+x^2}$	(z) $\frac{2}{\sqrt{1-x^2}}$	

2. Now try these. Think it through.

(a) $x^8 + 2$	(b) $e^{8x} - c$	(c) $-2 \sin 4x$	(d) $\cos 2\theta - \sec^2 2\theta$	
(e) $x^2 + 3x - 1$	(f) $\frac{\cos x}{2}$	(g) $x^{5/2}$	(h) $x^{-3/5}$	(i) $\frac{1}{\sqrt[4]{x^7}}$
(j) $\frac{4x^3}{x^4 + 12}$	(k) $14x(x^2 + 5)^6$	(l) $e^x \cos(e^x)$	(m) $12xe^{x^2+9}$	(n) $\frac{12(\ln x)^5}{x}$

Answers

1. Answers

(a) $x^7 + c$	(b) $\frac{x^7}{7} + c$	(c) $\frac{2}{7}x^7 + c$	(d) $e^x + c$	(e) $2e^x + c$
(f) $e^{2x} + c$	(g) $\frac{1}{2}e^{2x} + c$	(h) $\ln x + c$	(i) $-8 \ln x + c$	
(j) $\sin x + c$	(k) $4 \sin x + c$	(l) $\sin(4x) + c$	(m) $\frac{1}{4} \sin(4x) + c$	(n) $-\ln x + 4 \sin x + c$
(o) $-2 \sec x + c$	(p) $-\frac{1}{4} \cos 4x + c$	(q) $4 \tan x + c$	(r) $x^2 + c$	(s) $\frac{1}{2}x^2 + c$
(t) $4x^2 + c$	(u) $6^x + c$	(v) $3x^2 - 6^x + c$		
(w) $6x + c$	(x) $6x + \tan x + c$	(y) $6 \arctan(x) + c$	(z) $2 \arcsin(x) + c$	

2. Answers.

(a) $\frac{1}{9}x^9 + 2x + c$	(b) $\frac{1}{8}e^{8x} - cx + d$	(c) $\frac{1}{2} \cos 4x + c$	(d) $\frac{1}{2} \sin 2\theta - \frac{1}{2} \tan 2\theta + c$
(e) $\frac{1}{3}x^3 + \frac{3}{2}x^2 - x + c$	(f) $\frac{\sin x}{2} + c$	(g) $\frac{2}{7}x^{7/2} + c$	(h) $\frac{5}{2}x^{2/5} + c$
(i) $-\frac{4}{3}x^{-3/4}$	(j) $\ln(x^4 + 12) + c$	(k) $(x^2 + 5)^7 + c$	(l) $\sin(e^x) + c$
(m) $6e^{x^2+9} + c$	(n) $2(\ln x)^6 + c$		

General Antiderivative Rules

The key idea is that each derivative rule can be written as an antiderivative rule. We've seen how this works with specific functions like $\sin x$ and e^x and now we examine how the general derivative rules can be 'reversed.'

FACT 41.1 (Sum Rule). The sum rule for derivatives says

$$\frac{d}{dx}(F(x) \pm G(x)) = \frac{d}{dx}(F(x)) \pm \frac{d}{dx}(G(x)).$$

The corresponding antiderivative rule is

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx.$$

FACT 41.2 (Constant Multiple Rule). The constant multiple rule for derivatives says

$$\frac{d}{dx}(cF(x)) = c \frac{d}{dx}(F(x)).$$

The corresponding antiderivative rule is

$$\int cf(x) dx = c \int f(x) dx.$$

Examples

$$\int 8x^3 - 7\sqrt{x} dx = \int 8x^3 dx - \int 7x^{1/2} dx = 8 \int x^3 dx - 7 \int x^{1/2} dx = \frac{8x^4}{4} - \frac{7x^{3/2}}{3/2} + c = 2x^4 - \frac{14x^{3/2}}{3} + c$$

$$\int 6 \cos 2x - \frac{7}{x} + 2x^{-1/3} dx = 6 \int \cos 2x dx - 7 \int \frac{1}{x} dx + 2 \int x^{-1/3} dx = 6 \cdot \frac{1}{2} \cdot \sin 2x - 7 \ln |x| + \frac{2x^{2/3}}{2/3} + c = 3 \sin 2x - 7 \ln |x| + 3x^{2/3} + c.$$

$$\int 3e^{x/2} - \frac{8}{\sqrt{9-x^2}} dx = 3 \int e^{x/2} dx - 8 \int \frac{1}{\sqrt{9-x^2}} dx = 3 \cdot 2e^{x/2} + 8 \arcsin \frac{x}{3} + c = 6e^{x/2} + 8 \arcsin \frac{x}{3} + c.$$

Rewriting

Rewriting the integrand can greatly simplify the antiderivative process.

$$\int 2\sqrt[5]{t^2} - 6 \sec^2 t dt = \int 2t^{2/5} - 6 \sec^2 t dt = \frac{2t^{7/5}}{7/5} - 6 \tan t + c = \frac{10t^{7/5}}{7} - 6 \tan t + c.$$

$$\int \frac{x^4 + 2}{x^2} dx = \int x^2 + 2x^{-2} dx = \frac{x^3}{3} + \frac{2x^{-1}}{-1} + c = \frac{x^3}{3} - 2x^{-1} + c.$$

$$\int 6x^2(x^4 - 1) dx = \int 6x^6 - 6x^2 dx = \frac{6x^7}{7} + \frac{6x^3}{3} + c = \frac{6x^7}{7} + 2x^3 + c.$$

$$\int \frac{2}{\sqrt[3]{t^5}} dt = \int 2t^{-5/3} dt = \frac{2t^{-2/3}}{-2/3} + c = -3t^{-2/3} + c.$$

$$\int \frac{8x^2 + 7}{\sqrt{x}} dx = \int 8x^{3/2} + 7x^{-1/2} dx = \frac{8x^{5/2}}{5/2} + \frac{7x^{1/2}}{1/2} + c = \frac{16x^{5/2}}{5} + 14x^{1/2} + c.$$