Antiderivatives

40.1 Introduction

So far much of the term has been spent finding derivatives or rates of change. But in some circumstances we already know the rate of change and we wish to determine the original function. For example, meters or data loggers often measure rates of change, e.g., miles per hour or kilowatts per hour.

If you know the velocity of an object, can you determine the position of the object. This could happen in a car, say, where the speedometer readings were being recorded. Can the position of the car be determined from this information? Similarly, can the position of an airplane be determined from the black box which records the airspeed?

More generally, given $f'(x)$ can we find the function $f(x)$. If you think about it, this is the sort of question I have asked you to do on labs, tests, and homework assignments where I gave you the graph of $f'(x)$ and said draw the graph of $f(x)$. Or where I gave you the number line information for $f'(x)$ and $f''(x)$ and asked you to reconstruct the graph of $f(x)$. Can we do this same thing if we start with a formula for $f'(x)$? Can we get an explicit formula for $f(x)$? We usually state the problem this way.

DEFINITION 40.1. Let $f(x)$ be a function defined on an interval I. We say that $F(x)$ is an **antiderivative** of $f(x)$ on I if

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

EXAMPLE 40.1. If $f(x) = 2x$, then $F(x) = x^2$ is an antiderivative of f because

$$F'(x) = 2x = f(x).$$

But so is $G(x) = x^2 + 1$ or, more generally, $H(x) = x^2 + c$.

Are there 'other' antiderivatives of $f(x) = 2x$ besides those of the form $H(x) = x^2 + c$? We can use the MVT to show that the answer is 'No.' The proof will require three small steps.

THEOREM 40.1 (Theorem 1)**.** If $F'(x) = 0$ for all x in an interval I, then $F(x) = k$ is a constant function.

This makes a lot of sense: If the velocity of an object is 0, then its position is constant (not changing). Here's the

Proof. To show that $F(x)$ is constant, we must show that any two output values of F are the same, i.e., $F(a) = F(b)$ for all a and b in I.

So pick any a and b in I (with $a < b$). Then since F is differentiable on I, then F is both continuous and differentiable on the smaller interval $[a, b]$. So the MVT applies. There is a point c between a and b so that

$$\frac{F(b)-F(a)}{b-a} = F'(c) \Rightarrow F(b)-F(a) = F'(c)[b-a] = 0[b-a] = 0.$$

This means

$$F(b) = F(a),$$

in other words, F is constant. □

THEOREM 40.2 (Theorem 2)**.** Suppose that F, G are differentiable on the interval I and $F'(x) = G'(x)$ for all x in I. Then there exist k so that

$$G(x) = F(x) + k.$$

Proof. Consider the function $G(x) - F(x)$ on I. Then

$$\frac{d}{dx}\left(G(x) - F(x)\right) = G'(x) - F'(x) = 0.$$

Therefore, by Theorem 1

$$G(x) - F(x) = k$$

so

$$G(x) = F(x) + k.$$

□

THEOREM 40.3 (Theorem 3: Families of Antiderivatives)**.** If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval I, then $G(x) = F(x) + k$.

This is the theorem we want to show.

Proof. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval I then $G'(x) = f(x)$ and $F'(x) = f(x)$, that is, $F'(x) = G'(x)$ on I. Then by Theorem 2 $G(x) = F(x) + k$. □

DEFINITION 40.2. If $F(x)$ is any antiderivative of $f(x)$, we say that $F(x) + c$ is the **general antiderivative** of $f(x)$ on I.

Notation for Antiderivatives

Antidifferentiation is also called 'indefinite integration.'

$$\int f(x)\,dx = F(x) + c.$$

- $\int$ is the integration symbol
- $f(x)$ is called the integrand
- dx indicates the variable of integration
- $F(x)$ is a particular antiderivative of $f(x)$
- and c is the constant of integration.
- We refer to $\int f(x)\,dx$ as an 'antiderivative of $f(x)$' or an 'indefinite integral of f.'

Here are several examples.

$$\int 2x\,dx = x^2 + c$$

$$\int \cos t\,dt = \sin t + c$$

$$\int e^z\,dz = e^z + c$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + c$$

Antidifferentiation reverses differentiation so

$$\int F'(x)\,dx = F(x) + c$$

and differentiation undoes antidifferentiation

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Differentiation and antidifferentiation are reverse processes, so each derivative rule has a corresponding antidifferentiation rule.

Differentiation	**Antidifferentiation**
$\frac{d}{dx}(c) = 0$	$\int 0\,dx = c$
$\frac{d}{dx}(kx) = k$	$\int k\,dx = kx + c$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
$\frac{d}{dx}(\ln\|x\|) = \frac{1}{x}$	$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln\|x\| + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + c$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + c$
$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$
$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan x + c$

Variations and Generalizations

Notice what happens when we use ax instead of x in some of these functions. We multiply by a when taking the derivative, so we have to divide by a when taking the antiderivative.

Differentiation	**Antidifferentiation**
$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}(\sin ax) = a\cos ax$	$\int \cos ax\,dx = \frac{1}{a}\sin x + c$
$\frac{d}{dx}(\tan ax) = a\sec^2 ax$	$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c$
$\frac{d}{dx}(\arcsin(\frac{x}{a}) = \frac{1}{\sqrt{a^2-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin(\frac{x}{a}) + c$
$\frac{d}{dx}(\arctan(\frac{x}{a})) = \frac{a}{a^2+x^2}$	$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + c$

Try filling in the rules for $\int \sin ax\,dx$ and $\int \sec(ax)\tan(ax)\,dx$.

EXAMPLE 40.2. Here are a few examples.

$$\int \cos(4x)\,dx = \frac{1}{4}\sin(4x) + c$$
$$\int e^{z/2}\,dz = 2e^{z/22} + c$$
$$\int \frac{1}{16+x^2}\,dx = \frac{1}{4}\arctan\tfrac{x}{4} + c$$

40.2 Problems

1. Determine antiderivatives of the following functions. Take the derivative of your answer to confirm that you are correct. Why should you add $+c$ to any answer? **Basics**:

(a) $7x^6$ (b) x^6 (c) $2x^6$ (d) e^x (e) $2e^x$

(f) $2e^{2x}$ (g) e^{2x} (h) $\frac{1}{x}$ (i) $-\frac{8}{x}$

(j) $\cos x$ (k) $4\cos x$ (l) $4\cos(4x)$ (m) $\cos(4x)$ (n) $-\frac{1}{x} + 4\cos x$

(o) $-2\sec x\tan x$ (p) $\sin 4x$ (q) $4\sec^2 x$ (r) $2x$ (s) x

(t) $8x$ (u) $6^x\ln 6$ (v) $6x - 6^x\ln 6$

(w) 6 (x) $6 + \sec^2 x$ (y) $\frac{6}{1+x^2}$ (z) $\frac{2}{\sqrt{1-x^2}}$

2. Now try these. Think it through.

(a) $x^8 + 2$ (b) $e^{8x} - c$ (c) $-2\sin 4x$ (d) $\cos 2\theta - \sec^2 2\theta$

(e) $x^2 + 3x - 1$ (f) $\frac{\cos x}{2}$ (g) $x^{5/2}$ (h) $x^{-3/5}$ (i) $\frac{1}{\sqrt[4]{x^7}}$

(j) $\frac{4x^3}{x^4+12}$ (k) $14x(x^2+5)^6$ (l) $e^x\cos(e^x)$ (m) $12xe^{x^2+9}$ (n) $\frac{12(\ln x)^5}{x}$

Answers

1. Answers

(a) $x^7 + c$ (b) $\frac{x^7}{7} + c$ (c) $\frac{2}{7}x^7 + c$ (d) $e^x + c$ (e) $2e^x + c$

(f) $e^{2x} + c$ (g) $\frac{1}{2}e^{2x} + c$ (h) $\ln|x| + c$ (i) $-8\ln|x| + c$

(j) $\sin x + c$ (k) $4\sin x + c$ (l) $\sin(4x) + c$ (m) $\frac{1}{4}\sin(4x) + c$ (n) $-\ln|x| + 4\sin x + c$

(o) $-2\sec x + c$ (p) $-\frac{1}{4}\cos 4x + c$ (q) $4\tan x + c$ (r) $x^2 + c$ (s) $\frac{1}{2}x^2 + c$

(t) $4x^2 + c$ (u) $6^x + c$ (v) $3x^2 - 6^x + c$

(w) $6x + c$ (x) $6x + \tan x + c$ (y) $6\arctan(x) + c$ (z) $2\arcsin(x) + c$

2. Answers.

(a) $\frac{1}{9}x^9 + 2x + c$ (b) $\frac{1}{8}e^{8x} - cx + d$ (c) $\frac{1}{2}\cos 4x + c$ (d) $\frac{1}{2}\sin 2\theta - \frac{1}{2}\tan 2\theta + c$

(e) $\frac{1}{3}x^3 + \frac{3}{2}x^2 - x + c$ (f) $\frac{\sin x}{2} + c$ (g) $\frac{2}{7}x^{7/2} + c$ (h) $\frac{5}{2}x^{2/5} + c$

(i) $-\frac{4}{3}x^{-3/4}$ (j) $\ln(x^4+12) + c$ (k) $(x^2+5)^7 + c$ (l) $\sin(e^x) + c$

(m) $6e^{x^2+9} + c$ (n) $2(\ln x)^6 + c$

General Antiderivative Rules

The key idea is that each derivative rule can be written as an antiderivative rule. We've seen how this works with specific functions like $\sin x$ and e^x and now we examine how the general derivative rules can be 'reversed.'

FACT 41.1 (Sum Rule). The sum rule for derivatives says

$$\frac{d}{dx}(F(x) \pm G(x)) = \frac{d}{dx}(F(x)) \pm \frac{d}{dx}(G(x)).$$

The corresponding antiderivative rule is

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

FACT 41.2 (Constant Multiple Rule). The constant multiple rule for derivatives says

$$\frac{d}{dx}(cF(x)) = c\frac{d}{dx}(F(x)).$$

The corresponding antiderivative rule is

$$\int cf(x)\,dx = c\int f(x)\,dx.$$

Examples

$$\int 8x^3 - 7\sqrt{x}\,dx = \int 8x^3\,dx - \int 7x^{1/2}\,dx = 8\int x^3\,dx - 7\int x^{1/2}\,dx = \frac{8x^4}{4} - \frac{7x^{3/2}}{3/2} + c = 2x^4 - \frac{14x^{3/2}}{3} + c$$

$$\int 6\cos 2x - \frac{7}{x} + 2x^{-1/3}\,dx = 6\int \cos 2x\,dx - 7\int \frac{1}{x}\,dx + 2\int x^{-1/3}\,dx = 6\cdot\frac{1}{2}\cdot\sin 2x - 7\ln|x| + \frac{2x^{2/3}}{2/3} + c = 3\sin 2x - 7\ln|x| + 3x^{2/3} + c.$$

$$\int 3e^{x/2} - \frac{8}{\sqrt{9-x^2}}\,dx = 3\int e^{x/2}\,dx - 8\int \frac{1}{\sqrt{9-x^2}}\,dx = 3\cdot 2e^{x/2} + 8\arcsin\tfrac{x}{3} + c. = 6e^{x/2} + 8\arcsin\tfrac{x}{3} + c.$$

Rewriting

Rewriting the integrand can greatly simplify the antiderivative process.

$$\int 2\sqrt[5]{t^2} - 6\sec^2 t\,dt = \int 2t^{2/5} - 6\sec^2 t\,dt = \frac{2t^{7/5}}{7/5} - 6\tan t + c = \frac{10t^{7/5}}{7} - 6\tan t + c.$$

$$\int \frac{x^4+2}{x^2}\,dx = \int x^2 + 2x^{-2}\,dx = \frac{x^3}{3} + \frac{2x^{-1}}{-1} + c = \frac{x^3}{3} - 2x^{-1} + c.$$

$$\int 6x^2(x^4-1)\,dx = \int 6x^6 - 6x^2\,dx = \frac{6x^7}{7} + \frac{6x^3}{3} + c = \frac{6x^7}{7} + 2x^3 + c.$$

$$\int \frac{2}{\sqrt[3]{t^5}}\,dt = \int 2t^{-5/3}\,dt = \frac{2t^{-2/3}}{-2/3} + c = -3t^{-2/3} + c.$$

$$\int \frac{8x^2+7}{\sqrt{x}}\,dx = \int 8x^{3/2} + 7x^{-1/2}\,dx = \frac{8x^{5/3}}{5/3} + \frac{7x^{1/2}}{1/2} + c = \frac{24x^{5/3}}{5} + 14x^{1/2} + c.$$

41.3 Evaluating 'c' (Initial Value Problems)

So far we have been calculating general antiderivatives of functions. What this means is that if we know the velocity $v(t)$ of the car we are driving in, we can determine the position $p(t)$ of the car, up to a constant c if we can find an antiderivative for $v(t)$. If we have more information, the position of the car at a particular time say, then we are able to determine the precise antiderivative.

EXAMPLE 41.3. Suppose that $f'(x) = e^x + 2x$ and $f(0) = 3$. Find $f(x)$. $f(0)$ is sometimes called the *initial value* and such questions are referred to as *initial value problems*. [How could you interpret this information in terms of motion?]

SOLUTION. $f(x)$ must be an antiderivative of $f'(x)$ so

$$f(x) = \int f'(x)\,dx = \int e^x + 2x\,dx = e^x + x^2 + c.$$

Now use the initial value to solve for c:

$$f(0) = e^0 + 0^2 + c = 3 \Rightarrow 1 + c = 3 \Rightarrow c = 2.$$

Therefore, $f(x) = e^x + x^2 + 2$.

EXAMPLE 41.4. Suppose that $f'(x) = 6x^2 - 2x^3$ and $f(1) = 4$. Find $f(x)$.

SOLUTION. Again $f(x)$ must be an antiderivative of $f'(x)$ so

$$f(x) = \int f'(x)\,dx = \int 6x^2 - 2x^3\,dx = 2x^3 - \frac{x^4}{2} + c.$$

Now use the 'initial' value to solve for c:

$$f(1) = 2 - 1/2 + c = 3 \Rightarrow c = 2.5.$$

Therefore, $f(x) = 2x^3 - \frac{x^4}{2} + 2.5$.

EXAMPLE 41.5. Suppose that $f''(t) = 6t^{-2}$ (think acceleration) with $f'(1) = 8$ (think velocity) and $f(1) = 3$ (think position). Find $f(t)$.

SOLUTION. First find $f'(t)$ which must be the antiderivative of $f''(t)$. So

$$f'(t) = \int f''(t)\,dt = \int 6x^{-2}\,dt = -6t^{-1} + c.$$

Now use the 'initial' value for $f'(t)$ to solve for c:

$$f'(1) = -6(1) + c = 8 \Rightarrow c = 14.$$

Therefore, $f'(t) = -6t^{-1} + 14$. Now we are back to the earlier problem.

$$f(t) = \int f'(t)\,dt = \int -6t^{-1} + 14\,dt = -6\ln|t| + 14t + c.$$

Now use the 'initial' value of f to solve for c:

$$f(1) = -6\ln 1 + 14(1) + c = 3 \Rightarrow 6(0) + 14 + c = 3 \Rightarrow c = -11.$$

So $f(t) = 6\ln|t| + 14t - 11$.

In Class Practice

EXAMPLE 41.6. Find f given that $f'(x) = 6\sqrt{x} + 5x^{\frac{3}{2}}$ where $f(1) = 10$.

SOLUTION. $f(x)$ must be an antiderivative of $f'(x)$ so

$$f(x) = \int f'(x)\,dx = \int 6\sqrt{x} + 5x^{\frac{3}{2}}\,dx = 4x^{3/2} + 2x^{5/2} + c.$$

Use the 'initial' value to solve for c:

$$f(1) = 4 + 2 + c = 10 \Rightarrow c = 4.$$

Therefore, $f(x) = 4x^{3/2} + 2x^{5/2} + 4$.

EXAMPLE 41.7. Find f given that $f''(\theta) = \sin\theta + \cos\theta$ where $f'(0) = 1$ and $f(0) = 2$.

SOLUTION. First find $f'(\theta)$ which must be the antiderivative of $f''(\theta)$. So

$$f'(\theta) = \int f''(\theta)\,d\theta = \int \sin\theta + \cos\theta = -\cos\theta + \sin\theta + c.$$

Now use the initial value for $f'(\theta)$ to solve for c:

$$f'(0) = -\cos 0 + \sin 0 + c = -1 + 0 + c = 1 \Rightarrow c = 2.$$

Therefore, $f'(\theta) = -\cos\theta + \sin\theta + 2$.

$$f(\theta) = \int f'(\theta)\,d\theta = \int -\cos\theta + \sin\theta + 2\,d\theta = -\sin\theta - \cos\theta + 2\theta + c.$$

Now use the initial value of f to solve for c:

$$f(0) = -\sin 0 - \cos 0 + 2(0) + c = 0 - 1 + c = 2 \Rightarrow c = 3.$$

So $f(\theta) = -\sin\theta - \cos\theta + 2\theta + 3$.

Motion Problems

41.4 *Introduction*

Earlier in the term we interpreted the first and second derivatives as velocity and acceleration in the context of motion. So let's apply the initial value problem results to motion problems. Recall that

- $s(t) =$ position at time t.
- $s'(t) = v(t)$ velocity at time t.
- $s''(t) = v'(t) = a(t)$ acceleration at time t.

Therefore

- $\int a(t)\,dt = v(t) + c_1$ velocity.
- $\int v(t)\,dt = s(t) + c_2$ position at time t.

We will need to use additional information to evaluate the constants c_1 and c_2.

EXAMPLE 41.8. Suppose that the acceleration of an object is given by $a(t) = 2 - \cos t$ for $t \geq 0$ with

- $v(0) = 1$, this is also denoted v_0
- $s(0) = 3$, this is also denoted s_0.

Find $s(t)$.

SOLUTION. First find $v(t)$ which is the antiderivative of $a(t)$.

$$v(t) = \int a(t)\,dt = \int 2 - \cos t\,dt = 2t - \sin t + c_1.$$

Now use the initial value for $v(t)$ to solve for c_1:

$$v(0) = 0 - 0 + c_1 = 1 \Rightarrow c_1 = 1.$$

Therefore, $v(t) = 2t - \sin t + 1$. Now solve for $s(t)$ by taking the antiderivative of $v(t)$.

$$s(t) = \int v(t)\,dt = \int 2t - \sin t + 1\,dt = t^2 + \cos t + t + c_2$$

Now use the initial value of s to solve for c_2:

$$s(0) = 0 + \cos 0 + c = 3 \Rightarrow 1 + c = 3 \Rightarrow c = 2.$$

So $s(t) = t^2 + \cos t + 2t + 2$.

EXAMPLE 41.9. If acceleration is given by $a(t) = 10 + 3t - 3t^2$, find the position function if $s(0) = 1$ and $s(2) = 11$.

SOLUTION. First

$$v(t) = \int a(t)\,dt = \int 10 + 3t - 3t^2\,dt = 10t + \frac{3}{2}t^2 - t^3 + c.$$

Now

$$s(t) = \int 10t + \frac{3}{2}t^2 - t^3 + c\,dt = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 + ct + d.$$

But $s(0) = 0 + 0 - 0 + 0 + d = 1$ so $d = 1$. Then $s(2) = 20 + 4 - 4 + 2c + 1 = 11$ so $2c = -10 \Rightarrow c = -5$. Thus, $s(t) = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 - 5t + 1$.

EXAMPLE 41.10. If acceleration is given by $a(t) = \sin t + \cos t$, find the position function if $s(0) = 1$ and $s(2\pi) = -1$.

SOLUTION. First

$$v(t) = \int a(t)\,dt = \int \sin t + \cos t\,dt = -\sin t + \cos t + c.$$

Now

$$s(t) = \int -\sin t + \cos t + c\,dt = -\cos t - \sin t + ct + d.$$

But $s(0) = -1 + 0 - 0 + 0 + d = 1$ so $d = 2$. Then $s(\pi) = -1 + 0 + \pi c + 2 = -1$ so $2\pi c = -2 \Rightarrow c = -1/\pi$. Thus, $s(t) = -\cos t - \sin t - \frac{t}{\pi} + 2$.

41.5 *Constant Acceleration: Gravity*

In many motion problems the acceleration is constant. This happens when an object is thrown or dropped and the only acceleration is due to gravity. In such a situation we have

- $a(t) = a$, constant acceleration
- with initial velocity $v(0) = v_0$
- and initial position $s(0) = s_0$.

Then

$$v(t) = \int a(t)\,dt = \int a\,dt = at + c.$$

But

$$v(0) = a \cdot 0 + c = v_0 \Rightarrow c = v_0.$$

So

$$\boxed{v(t) = at + v_0.}$$

Next,

$$s(t) = \int v(t)\,dt = \int at + v_0\,dt = \tfrac{1}{2}at^2 + v_0 t + c.$$

At time $t = 0$,

$$s(0) = \tfrac{1}{2}a(0)^2 + v_0(0) + c = s_0 \Rightarrow c = s_0.$$

Therefore

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0.}$$

EXAMPLE 41.11. Suppose a ball is thrown with initial velocity 96 ft/s from a roof top 432 feet high. The acceleration due to gravity is constant $a(t) = -32$ ft/s^2. Find $v(t)$ and $s(t)$. Then find the maximum height of the ball and the time when the ball hits the ground.

SOLUTION. Recognizing that $v_0 = 96$ and $s_0 = 432$ and that the acceleration is constant, we may use the general formulas we just developed.

$$\boxed{v(t) = at + v_0 = -32t + 96}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -16t^2 + 96t + 432.}$$

The max height occurs when the velocity is 0:

$$v(t) = -32t + 96 = 0 \Rightarrow t = 3 \Rightarrow s(3) = -144 + 288 + 432 = 576.$$

The ball hits the ground when $s(t) = 0$.

$$s(t) = -16t^2 + 96t + 432 = -16(t^2 - 6t - 27) = -16(t-9)(t+3) = 0.$$

So $t = 9$ and $(t \neq -3)$.

YOU TRY IT 41.1. A stone is thrown upward with an initial velocity of 48ft/s from the edge of a cliff 64ft above a river. (Remember acceleration due to gravity is -32ft/s^2.)

(o) Find the velocity of the stone for $t \geq 0$.

(p) Find the position of the stone for $t \geq 0$.

(q) Find the time when it reaches its highest point (and the height).

(r) Find the time when the stone hits the ground.

Answers: $v(t) = -32t + 48$, $s(t) = -16t^2 + 48t + 64$, max ht: 100 ft at $t = 1.5$s, hits ground at $t = 4$s.

EXAMPLE 41.12. A person drops a stone from a bridge. What is the height (in feet) of the bridge if the person hears the splash 5 seconds after dropping it?

SOLUTION. Here's what we know. $v_0 = 0$ (dropped) and $s(5) = 0$ (hits water). And we know acceleration is constant, $a = -32$ ft/s^2. We want to find the height of the bridge, which is just s_0. Use our constant acceleration motion formulas to solve for a.

$$\boxed{v(t) = at + v_0 = -32t}$$

and

$$\boxed{s(t) = \frac{1}{2}at^2 + v_0t + s_0 = -16t^2 + s_0.}$$

Now we use the position we know: $s(5) = 0$.

$$s(5) = -16(5)^2 + s_0 \Rightarrow s_0 = 400.$$

Notice that we did not need to use the velocity function.

YOU TRY IT 41.2 (Extra Credit). In the previous problem did you take into account that sound does not travel instantaneously in your calculation above? Assume that sound travels at 1120 ft/s. What is the height (in feet) of the bridge if the person hears the splash 5 seconds after dropping it?

EXAMPLE 41.13. Here's a variation and this time we will use metric units. Suppose a ball is thrown with unknown initial velocity v_0 ft/s from a roof top 49 meter high and the position of the ball at time $t = 3$ s is $s(3) = 0$. The acceleration due to gravity is constant $a(t) = -9,8$ m/s^2. Find $v(t)$ and $s(t)$.

SOLUTION. This time v_0 is unknown but $s_0 = 49$ and $s(3) = 0$. Again the acceleration is constant so we may use the general formulas for this situation.

$$v(t) = at + v_0 = -9.8t + v_0$$

and

$$s(t) = \frac{1}{2}at^2 + v_0t + s_0 = -4.9t^2 + v_0t + 49.$$

But we know that

$$s(3) = -4.9(3)^2 + v_0(3) + 49 = 0$$

which means

$$3v_0 = 4.9(9) - 4.9(10) = -4.9 \Rightarrow v_0 = -4.9/3.$$

So

$$\boxed{v(t) = -9.8t - \frac{49}{30}}$$

and

$$\boxed{s(t) = -4.9t^2 - \frac{49}{30}t + 49.}$$

Interpret $v_0 = -4.9/3$.

EXAMPLE 41.14. Mo Green is attempting to run the 100m dash in the Geneva Invitational Track Meet in 9.8 seconds. He wants to run in a way that his *acceleration* is constant, a, over the entire race. Determine his velocity function? (a will still appear as an unknown constant.) Determine his position function? There should be no unknown constants in your equation. What is his velocity at the end of the race? Do you think this is realistic?

SOLUTION. We have: constant acc $= a\,\text{m/s}^2$; $v_0 = 0$ m/s; $s_0 = 0$ m. So

$$v(t) = at + v_0 = at$$

and

$$s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = \tfrac{1}{2}at^2.$$

But $s(9.8) = \frac{1}{2}a(9.8)^2 = 100$, so $a = \frac{200}{(9.8)^2} = 2.0825\,\text{m/s}^2$. So $s(t) = 2.0825t^2$. Mo's velocity at the end of the race is $v(9.8) = a(9.8) = 2.0825(9.8) = 20.41$ m/s...not realistic.

EXAMPLE 41.15. A stone dropped off a cliff hits the ground with speed of 120 ft/s. What was the height of the cliff?

SOLUTION. Recognizing that $v_0 = 0$ and s_0 is unknown and is the cliff height and that the acceleration is constant $a = -32$ ft/s, we may use the general formulas for motion with constant acceleration:

$$\boxed{v(t) = at + v_0 = -32t + 0 = -32t}.$$

Now we use the velocity and the one velocity we know $v = -120$ when it hits the ground so

$$v(t) = -32t = -120 \Rightarrow t = 120/32 = 15/4$$

when it hits the ground. Now remember when it hits the ground the height is 0. So $s(15/4) = 0$. But we know

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -16t^2 + 0t + s_0 = -16t^2 + s_0.}$$

Now substitute in $t = 15/4$ and solve for s_0.

$$s(15/4) = 0 \Rightarrow -16(15/4)^2 + s_0 = 0 \Rightarrow s_0 = 15^2 = 225.$$

The cliff height is 225 feet.

EXAMPLE 41.16. A car is traveling at 90 km/h when the driver sees a deer 75 m ahead and slams on the brakes. What constant deceleration is required to avoid hitting Bambi? [Note: First convert 90 km/h to m/s.]

SOLUTION. Let's list all that we know. $v_0 = 90$ km/h or $\frac{9000}{60\cdot 60} = 25$ m/s. Let $s_0 = 0$ and let time t^* represent the time it takes to stop. Then $s(t^*) = 75$ m. Now the car is stopped at time t^*, so we know $v(t^*) = 0$. Finally we know acceleration is an unknown constant, a, which is what we want to find.

Now we use our constant acceleration motion formulas to solve for a.

$$\boxed{v(t) = at + v_0 = at + 25}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = \tfrac{1}{2}at^2 + 25t.}$$

Now we use other velocity and position we know: $v(t^*) = 0$ and $s(t^*) = 75$ when the car stops. So

$$v(t^*) = at^* + 25 = 0 \Rightarrow t^* = -25/a$$

and

$$s(t^*) = \tfrac{1}{2}a(t^*)^2 + 25t^* = \tfrac{1}{2}a(-25/a)^2 + 25(-25/a) = 75.$$

Simplify to get

$$\frac{625a}{2a^2} - \frac{625}{a} = \frac{625}{2a} - \frac{1350}{2a} = -\frac{625}{2a} = 75 \Rightarrow 150a = -625$$

so

$$a = -\frac{625}{150} = -\frac{25}{6}\ \text{m/s}.$$

The time it takes to stop is

$$t^* = \frac{-25}{a} = \frac{-25}{-\frac{25}{6}} = 6.$$

EXAMPLE 41.17. One car intends to pass another on a back road. What constant acceleration is required to increase the speed of a car from 30 mph (44 ft/s) to 50 mph ($\frac{220}{3}$ ft/s) in 5 seconds?

SOLUTION. Given: $a(t) = a$ constant. $v_0 = 44$ ft/s. $s_0 = 0$. And $v(5) = \frac{220}{3}$ ft/s. Find a. But $v(t) = at + v_0 = at + 44$. So $v(5) = 5a + 44 = \frac{220}{3} \Rightarrow 5a = \frac{220}{3} - 44 = \frac{88}{3}$. Thus $a = \frac{88}{15}$.