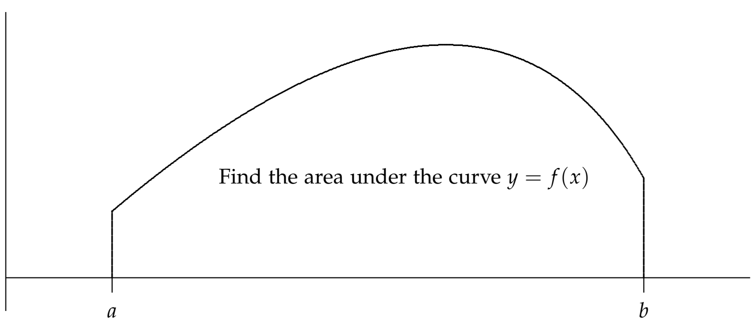
Calculus II: Preview Area Under Curves

So far we have interpreted antidifferentiation as "undoing" or "reversing" differentiation. But historically, antiderivatives or integrals arose in another way. As with derivatives where there is a geometric problem (the slope problem) that is solved, there is a geometric problem that is solved by integration.

The Area Problem

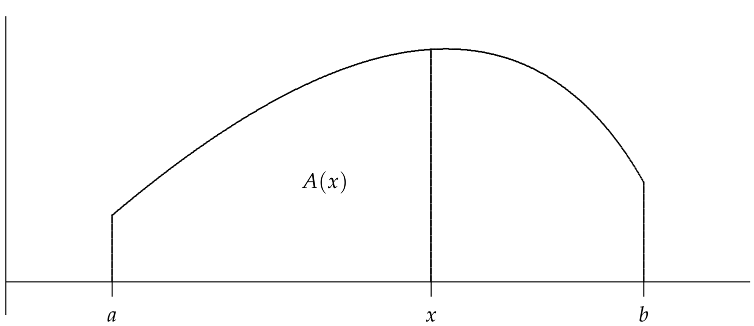
Let f be a continuous (nonnegative) curve on the closed interval $[a, b]$. Find the area bounded by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.



A Solution

Let f be as above. Suppose we define an 'area' function $A(x)$ on the closed interval $[a, b]$. In particular, for any x in $[a, b]$ define

$$A(x) = \text{the area under } f \text{ from } a \text{ to } x.$$



Notice that:

- $A(b)$ is the entire area under f from a to b and is the solution to the 'area problem.'

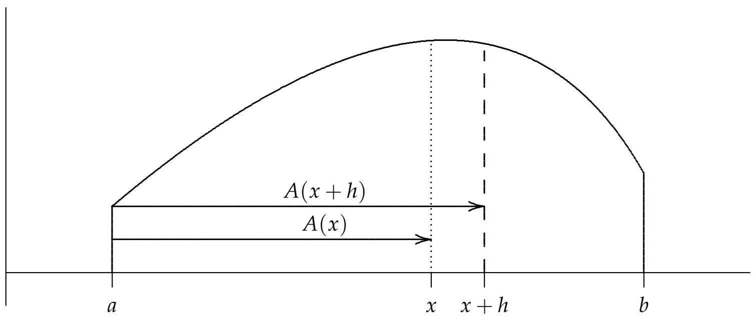
- $A(a) = 0$ since it represents the area from a to a under f.

Amazingly, while we don't even have a formula for $A(x)$, we can show A is differentiable! (That, of course, means that A is also continuous.) To do this, we go back to the definition of the derivative. This is good review for the final exam.

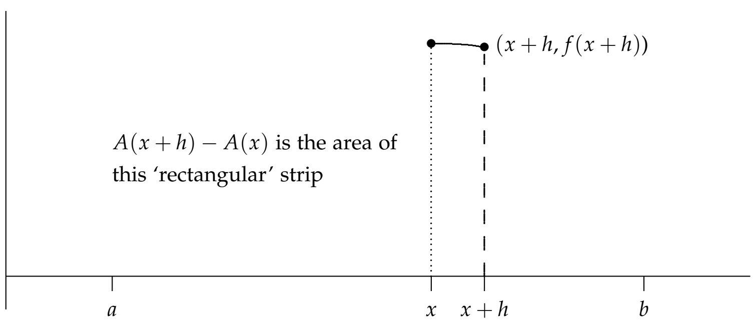
Recall that the definition of the derivative is

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}.$$

Now $A(x+h) =$ the area under f from a to $x+h$ (see the diagram).
And $A(x) =$ the area under f from a to x (see the diagram).



So the difference $A(x+h) - A(x)$ represents the area of the (nearly) rectangular strip between the dotted and dashed vertical lines.



The height of the 'rectangular' strip is $f(x+h)$ and the width of the strip is

$$(x+h) - x = h.$$

So the area of the strip is approximately $f(x+h) \cdot h$. Let's put all of this together.

$$\begin{aligned} A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} &\approx \lim_{h \to 0} \frac{\text{Area of 'rectangular' strip}}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) \cdot h}{h} \\ &= \lim_{h \to 0} f(x+h) \\ &= f(x), \end{aligned}$$

where the last equality follows because we assumed that f was a continuous function. Look at what this means:

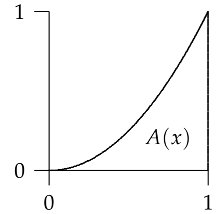
$$A'(x) = f(x);$$

in other words, the derivative of the area function is the original curve! Said differently, $A(x)$ is an antiderivative of $f(x)$, that is

$$\text{Area from } a \text{ to } x = A(x) = \int f(x)\, dx.$$

Amazing Stuff!!

EXAMPLE 41.1. Here's why I say this is amazing. We know a few area formulas: rectangles, circles, triangles. But do you know how to find the area under something as simple as a quadratic curve? So here's the problem: Find the area under $y = f(x) = x^2$ on the interval $[0, 1]$.



SOLUTION. Let $A(x)$ be the area from 0 to x. We know that

$$A(x) = \int f(x)\,dx = \int x^2\,dx = \tfrac{1}{3}x^3 + c.$$

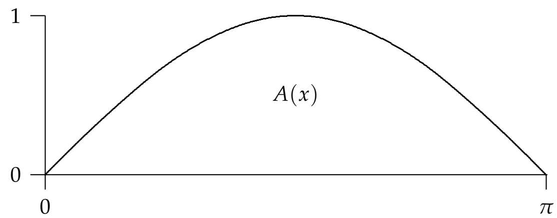
But we can determine c since we know the 'initial value' $A(0) = 0$ (why?). So

$$A(0) = \tfrac{1}{3}(0)^3 + c = 0 \Rightarrow c = 0.$$

Therefore $A(x) = \frac{1}{3}x^3$ and, in particular, $A(1) = \frac{1}{3}(1)^3 = \frac{1}{3}$.

What would $A(6)$ represent and what is its value? **Extra Credit.** With $f(x) = x^2$, what is that area under f between a and b?

EXAMPLE 41.2. Find the area under under $f(x) = \sin x$ on the interval $[0, \pi]$.



SOLUTION. Let $A(x)$ be the area from 0 to x. We know that

$$A(x) = \int f(x)\,dx = \int \sin x\,dx = -\cos x + c.$$

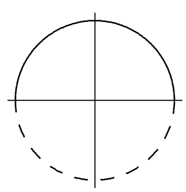
We can determine c since we know the 'initial value' $A(0) = 0$ again. So

$$A(0) = -\cos x + c = -1 + c = 0 \Rightarrow c = 1.$$

Therefore $A(x) = -\cos x + 1$ and, in particular, $A(\pi) = -\cos(\pi) + 1 = -(-1) + 1 = 2$. This is amazing, because by symmetry, this means that the area from 0 to $\pi/2$ is exactly 1. We could also verify this by using:

$$A(\pi/2) = -\cos(\pi/2) + 1 = -(0) + 1 = 1.$$

EXAMPLE 41.3. Since the equation of the unit circle is $x^2 + y^2 = 1$, the equation of the upper unit semi-circle is $y = \sqrt{1 - x^2}$. Find the area under under it on the interval $[-1, 1]$.



SOLUTION. Let $A(x)$ be the area from -1 to x. We know that

$$A(x) = \int f(x)\,dx = \int \sqrt{1 - x^2}\,dx = ???$$

We're stuck! So we need to learn how to do more antiderivatives. Take Calculus II. There you will prove that the area of a circle of radius r is πr^2 by doing a similar antidifferentiation.

But Wait, There's more!

Earlier we interpreted the antiderivative of velocity as position. But now we have seen that an antiderivative of $f(x)$ represents the area under the curve. So if we graph a velocity function over a time interval $[a, b]$, then the area under the curve on this interval is the (net) distance the car travels.

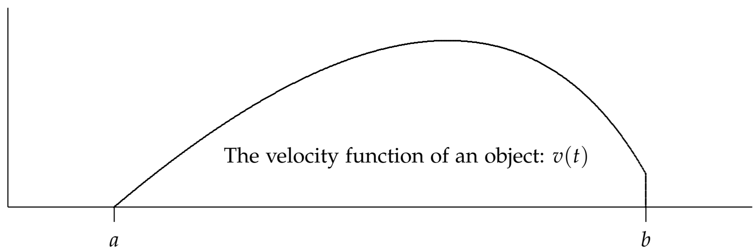


Figure 41.1: If we graph the velocity $v(t)$ of an object moving along a straight line, then the area under the velocity curve represents the distance travelled by the object during the corresponding time interval.