4.0 5-Minute Review: Rational Functions

DEFINITION. A **rational function**[1] is a function of the form

$$y = r(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials. The domain of a rational function consists of all values of x such that $q(x) \neq 0$.

[1] Here the term 'rational' means 'ratio' as in the ratio of two polynomials.

EXAMPLE 4.1. Here are several examples.

(a) $r(x) = \dfrac{2x^2+3x+1}{4x^{11}+9x^2}$ (b) $s(x) = \dfrac{1}{2x+7}$ (c) $t(x) = \dfrac{3x+1}{x^2+4} - \dfrac{2x}{4x+10}$

(d) $p(x) = 6x^3+1$ (e) $r(t) = 5t + t^{-2}$

And some non-examples (why aren't these rational functions?):

(a) $r(x) = \dfrac{\sin(2x^2+1)}{4x^{11}+9x^2}$ (b) $s(x) = \sqrt{\dfrac{3x}{2x+7}}$ (c) $t(x) = \dfrac{3x^{1/2}+1}{x^2+4}$

Caution: Often times we will need to simplify expressions involving rational functions. Be careful. Do you see why

$$r(x) = \frac{x^2-4}{x^2-2x} = \frac{(x-2)(x+2)}{x(x-2)}$$

and

$$s(x) = \frac{x+2}{x}$$

are *not* the same function? The domain of $r(x)$ is all $x \neq 0, 2$ while the domain of $s(x)$ is all $x \neq 0$. The two functions have the same values at all points where *both* are defined.

YOU TRY IT 4.1. Solve the following.

(a) Determine all x for which $\dfrac{x^2-4x-5}{x^2-1}$ and $\dfrac{x-5}{x-1}$ have the same values.

(b) Do the same for $\dfrac{x^3-3x^2+2x}{x^2-x}$ and $x-2$

(c) Carefully sketch the graph of $\dfrac{x^3-3x^2+2x}{x^2-x}$. Think about part (b).

4.1 5-Minute Review: Conjugates

Conjugates are usually discussed in reference to expressions involving square roots and typically have the form $a+\sqrt{b}$ and $a-\sqrt{b}$, where a can be any expression. For example, $\sqrt{x}+\sqrt{3}$ and $\sqrt{x}-\sqrt{3}$. Conjugates are useful because when you multiply them together, the 'middle terms' cancel: e.g.,

$$(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b$$

or

$$(\sqrt{x}+\sqrt{3})(\sqrt{x}-\sqrt{3}) = x-3.$$

Simplify each of these expressions by multiplying both the numerator and denominator by an appropriate conjugate.

(a) $\dfrac{x-5}{\sqrt{x}-\sqrt{5}}$ (b) $\dfrac{4}{\sqrt{x+2}-\sqrt{x}}$ (c) $\dfrac{2x-18}{\sqrt{x}-3}$ (d) $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

SOLUTION. For part (b) use

$$\frac{4}{\sqrt{x+2}-\sqrt{x}} \cdot \frac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}+\sqrt{x}} = \frac{4(\sqrt{x+2}+\sqrt{x})}{x+2-x} = \frac{4(\sqrt{x+2}+\sqrt{x})}{2} = 2(\sqrt{x+2}+\sqrt{x})$$

Calculating Limits

4.2 *Basic Limit Properties*

There are several basic limit properties which ease the problem of calculating limits. Most of these properties are quite straightforward and would be what you might suspect is true. In more advanced calculus courses (called analysis courses) each of these properties would be carefully proven.

The first property that we list is almost a tautology. It concerns the linear function $f(x) = x$.

THEOREM 4.1. $\lim_{x \to a} x = a$.

The limit is asking: 'As x approaches a, what is $y = f(x) = x$ approaching?' Well, obviously a! This is also geometrically clear in the left-hand panel of Figure 4.1.

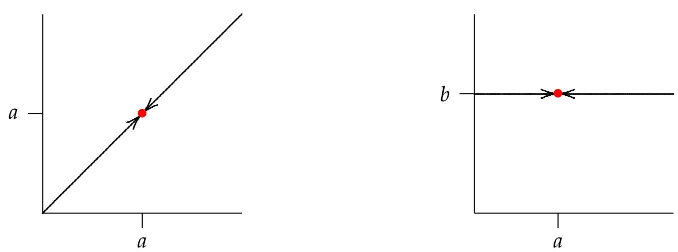

Figure 4.1: Left: 'As x approaches a, x approaches a' so $\lim_{x \to a} x = a$. Right: For 'As x approaches a, the constant function $f(x) = b$ is always equal to b' so $\lim_{x \to a} b = b$.

THEOREM 4.2. If b is any constant, then $\lim_{x \to a} b = b$.

The limit is asking: 'As x approaches a, what is $y = f(x) = b$ approaching?' Well, since b is constant, not is $f(x)$ approaching b, it is actually equal to b. This is geometrically clear in the right-hand panel of Figure 4.1.

The limit is asking: 'As x approaches a, what is $y = f(x) = b$ approaching?' Well, since b is constant, not is $f(x)$ approaching b, it is actually equal to b. This is geometrically clear in the right-hand panel of Figure 4.1.

We can combine these two theorems in a more general result about lines or linear functions.

THEOREM 4.3 (Linear functions)**.** Let a, b, and m be any constants and let f be the linear function $f(x) = mx + b$. Then

$$\lim_{x \to a} f(x) = \lim_{x \to a} mx + b = ma + b = f(a).$$

Remember from Lab 1, that a function where $\lim_{x \to a} f(x) = f(a)$ is called a **continuous** function at a.

Think about what this says: as $x \to a$, $mx \to ma$, so $mx + b \to ma + b$. We have actually used two mathematical operations, multiplication and addition, and said that the limit operation interacts with them in a special way: the order of operations does not matter. The results in the next section this clear in a number of situations.

EXAMPLE 4.2. Determine the following limits.

(*a*) $\lim_{x\to 4} f(x)$ where $f(x) = -2x + \frac{1}{2}$.

(*b*) $\lim_{t\to -2} g(t)$ where $g(t) = \frac{7}{2}t - 8$.

(*c*) $\lim_{x\to -1} h(x)$ where $h(x) = 9$.

SOLUTION. (*a*) $\lim_{x\to 4} f(x) = \lim_{x\to 4}\left(-2x + \frac{1}{2}\right) \overset{\text{Linear}}{=} f(4) = \dfrac{15}{2}$.

(*b*) $\lim_{t\to -2} g(t) = \lim_{t\to -2}\left(\frac{7}{2}t - 8\right) \overset{\text{Linear}}{=} g(-2) = -15$.

(*c*) $\lim_{x\to -1} h(x) = \lim_{x\to -1} 9 \overset{\text{Linear}}{=} 9$. (Here $m = 0$ or we could use the constant theorem.)

Properties Involving the Order of Operations

The next several properties involve the interchange of the limit operation with other arithmetic operations such as addition or multiplication. These properties are important because they greatly simplify the calculation of limits. It is important to recognize that not all mathematical operations can be interchanged in this way. For example, the square root of a sum is not the sum of the square roots:

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}.$$

In particular, $\sqrt{9+16} = 5$ while $\sqrt{9} + \sqrt{16} = 7$. Clearly, the order of operations matters; taking the square root before or after summing changes the result.

However, the limit properties below indicate that certain mathematical operations with limits will give the same answer, regardless of the order in which they are carried out. This is special and useful! Note the order of operations in each part.

THEOREM 4.4 (Order of Operations)**.** Assume that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ and that b is a constant. Then

1. **(Constant Multiple)**. $\lim_{x\to a} bf(x) = b(\lim_{x\to a} f(x)) = bL$.
2. **(Sum or Difference)**. $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$. 'The limit of a sum is the sum of the limits.'
3. **(Product)**. $\lim_{x\to a} f(x)g(x) = (\lim_{x\to a} f(x)) \cdot (\lim_{x\to a} g(x)) = LM$. 'The limit of a product is the product of the limits.'
4. **(Quotient)**. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, as long as $M \neq 0$. 'The limit of a quotient is the quotient of the limits.'

EXAMPLE 4.3. Determine the following limits. Indicate which limit properties were used at each step. Assume that $\lim_{x\to 2} f(x) = 4$ and $\lim_{x\to 2} g(x) = -3$. Evaluate

(*a*) $\lim_{x\to 2}[2f(x) + 3g(x)] \overset{\text{Sum}}{=} \lim_{x\to 2} 2f(x) + \lim_{x\to 2} 3g(x) \overset{\text{Const Mult}}{=} 2\lim_{x\to 2} f(x) + 3\lim_{x\to 2} g(x) = 2(4) + 3(-3) = -1$.

(*b*) $\lim_{x\to 2} -2f(x)g(x) \overset{\text{Prod}}{=} \lim_{x\to 2} -2f(x) \cdot \lim_{x\to 2} g(x) \overset{\text{Const Mult}}{=} -2\lim_{x\to 2} f(x) \cdot \lim_{x\to 2} g(x) = -2(4) \cdot (-3) = 24$.

(*c*) $\lim_{x\to 2} \dfrac{6x + 7 - f(x)}{g(x)} \overset{\text{Diff, Quot}}{=} \dfrac{\lim_{x\to 2}(6x+7) - \lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)} \overset{\text{Linear}}{=} \dfrac{19-4}{-3} = -5$.

(*d*) $\lim_{x\to a} x^2 = \lim_{x\to a} x\cdot x \overset{\text{Prod}}{=} \lim_{x\to a} x \cdot \lim_{x\to a} x \overset{\text{Thm 1}}{=} a\cdot a = a^2$.
More generally, if n is a positive integer, then $\lim_{x\to a} x^n = a^n$ by using the product limit law. That means the the function $f(x) = x^n$ is *continuous* for every point a.

THEOREM 4.5 (Order of Operations, Continued). Assume that $\lim_{x\to a} f(x) = L$ and that m and n are positive integers. Then

5. **(Power)**. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n = L^n$.
6. **(Fractional Power)**. Assume that $\frac{n}{m}$ is reduced. Then

$$\lim_{x\to a}[f(x)]^{n/m} = \left[\lim_{x\to a} f(x)\right]^{n/m} = L^{n/m},$$

provided that $f(x) \geq 0$ for x near a if m is even.

EXAMPLE 4.4. Determine $\lim_{x\to 3}(4x-1)^5$. Indicate which limit properties were used at each step.

SOLUTION.

$$\begin{aligned}\lim_{x\to 3}(4x-10)^5 &\overset{\text{Powers}}{=} [\lim_{x\to 3} 4x - 10]^5 \overset{\text{Diff}}{=} [\lim_{x\to 3} 4x - \lim_{x\to 3} 10]^5 \\ &\overset{\text{Const Mult}}{=} [4\lim_{x\to 3} x - \lim_{x\to 3} 10]^5 \\ &\overset{\text{Thm 5.1,5.2}}{=} [4(3)-10]^5 = (2)^5 = 32.\end{aligned}$$

EXAMPLE 4.5. Determine $\lim_{x\to 2}\sqrt{x^2-1}$. Indicate which limit properties were used at each step.

SOLUTION. Notice that $\sqrt{x^2-1} = (x^2-1)^{1/2}$ is a fractional power function. In the language of Theorem 4.5, $\frac{n}{m} = \frac{1}{2}$ is reduced and $m = 2$ is even. Near $x = 2$, $f(x) = x^2 - 1$ is positive. So Theorem 4.5 applies and we may calculate the limit as

$$\lim_{x\to 2}\sqrt{x^2-1} \overset{\text{Frac Pow}}{=} \left(\lim_{x\to 2} x^2 - 1\right)^{1/2} \overset{\text{Poly}}{=} (3)^{1/2} = \sqrt{3}.$$

EXAMPLE 4.6. Determine $\lim_{x\to -3}(2x^2 - x + 1)^{4/3}$. Indicate which limit properties were used at each step.

SOLUTION. $(2x^2 - x + 1)^{4/3}$ is a fractional power function with $\frac{n}{m} = \frac{4}{3}$ which is reduced and $m = 3$ is odd. Near $x = 2$, $f(x) = 2x^2 - x + 1$ is positive. So Theorem 4.5 applies and we may calculate the limit as

$$\lim_{x\to -3}\sqrt{2x^2-x+1} \overset{\text{Frac Pow}}{=} \left(\lim_{x\to -3} 2x^2 - x + 1\right)^{1/2} \overset{\text{Sum, Prod}}{=} (18 - 3 + 1)^{1/2} = 4.$$

EXAMPLE 4.7. Determine $\lim_{x\to -3}(x^2 - 25)^{3/4}$.

SOLUTION. $(x^2-25)^{3/4}$ is a fractional power function. In the language of Theorem 4.5, $\frac{n}{m} = \frac{3}{4}$ is reduced and $m = 4$ is even. Near $x = -3$, $x^2 - 25$ is *negative*. Since $(x^2-25)^{3/4}$ is not even defined near -3, this limit does not exist.