More Properties of Limits: Order of Operations

THEOREM 4.5 (Order of Operations, Continued). Assume that $\lim_{x\to a} f(x) = L$ and that m and n are positive integers. Then

5. **(Power)**. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n = L^n$.

6. **(Fractional Power)**. Assume that $\frac{n}{m}$ is reduced. Then

$$\lim_{x\to a}[f(x)]^{n/m} = \left[\lim_{x\to a} f(x)\right]^{n/m} = L^{n/m},$$

provided that $f(x) \geq 0$ for x near a if m is even.

EXAMPLE 4.4. Determine $\lim_{x\to 3}(4x-1)^5$. Indicate which limit properties were used at each step.

SOLUTION.

$$\begin{aligned}\lim_{x\to 3}(4x-10)^5 \overset{\text{Powers}}{=} [\lim_{x\to 3} 4x - 10]^5 &\overset{\text{Diff}}{=} [\lim_{x\to 3} 4x - \lim_{x\to 3} 10]^5 \\ &\overset{\text{Const Mult}}{=} [4\lim_{x\to 3} x - \lim_{x\to 3} 10]^5 \\ &\overset{\text{Thm 5.1,5.2}}{=} [4(3)-10]^5 = (2)^5 = 32.\end{aligned}$$

EXAMPLE 4.5. Determine $\lim_{x\to 2} \sqrt{x^2-1}$. Indicate which limit properties were used at each step.

SOLUTION. Notice that $\sqrt{x^2-1} = (x^2-1)^{1/2}$ is a fractional power function. In the language of Theorem 4.5, $\frac{n}{m} = \frac{1}{2}$ is reduced and $m = 2$ is even. Near $x = 2$, $f(x) = x^2 - 1$ is positive. So Theorem 4.5 applies and we may calculate the limit as

$$\lim_{x\to 2} \sqrt{x^2-1} \overset{\text{Frac Pow}}{=} \left(\lim_{x\to 2} x^2 - 1\right)^{1/2} \overset{\text{Poly}}{=} (3)^{1/2} = \sqrt{3}.$$

EXAMPLE 4.6. Determine $\lim_{x\to -3}(2x^2-x+1)^{4/3}$. Indicate which limit properties were used at each step.

SOLUTION. $(2x^2-x+1)^{4/3}$ is a fractional power function with $\frac{n}{m} = \frac{4}{3}$ which is reduced and $m = 3$ is odd. Near $x = 2$, $f(x) = 2x^2 - x + 1$ is positive. So Theorem 4.5 applies and we may calculate the limit as

$$\lim_{x\to -3} \sqrt{2x^2-x+1} \overset{\text{Frac Pow}}{=} \left(\lim_{x\to -3} 2x^2 - x + 1\right)^{1/2} \overset{\text{Sum, Prod}}{=} (18-3+1)^{1/2} = 4.$$

EXAMPLE 4.7. Determine $\lim_{x\to -3}(x^2-25)^{3/4}$.

SOLUTION. $(x^2-25)^{3/4}$ is a fractional power function. In the language of Theorem 4.5, $\frac{n}{m} = \frac{3}{4}$ is reduced and $m = 4$ is even. Near $x = -3$, $x^2 - 25$ is *negative*. Since $(x^2-25)^{3/4}$ is not even defined near -3, this limit does not exist.

We now look at some special cases of limits with familiar functions.

THEOREM 4.6 (Special Functions). Let n be a positive integer and c be any constant.

7. **(Monomials)**. $\lim_{x\to a} cx^n = ca^n$.

8. **(Polynomials)**. If $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ is a degree n polynomial, then

$$\lim_{x\to a} p(x) = p(a).$$

9. (Rational Functions). If $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function, then for any point a in the domain of $r(x)$

$$\lim_{x\to a} r(x) = r(a).$$

Theorem 4.6 says that the limit of polynomial or rational function as $x \to a$ is the same as the value of the function at $x = a$. This is not true of all limits. For example, we saw that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, yet we can't even put $x = 0$ into this function! Those special or 'nice' functions where $\lim_{x\to a} f(x) = f(a)$ are called **continuous** at $x = a$. We will examine them in depth in a few days. For the moment we can say that polynomials are continuous everywhere and rational functions are continuous at every point in their domains.

Proof. Let's see how limit properties 7 through 9 follow from the previous properties of limits. To prove the monomial property, use

$$\lim_{x\to a} cx^n \overset{\text{Const Mult}}{=} c[\lim_{x\to a} x^n] \overset{\text{Powers}}{=} c[\lim_{x\to a} x]^n \overset{\text{Thm 5.1}}{=} ca^n.$$

To prove the polynomial property, since $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ is a degree n polynomial, then

$$\begin{aligned}
\lim_{x\to a} p(x) &= \lim_{x\to a}[c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0] \\
&\overset{\text{Sum}}{=} \lim_{x\to a} c_n x^n + \lim_{x\to a} c_{n-1}x^{n-1} + \cdots + \lim_{x\to a} c_1 x + \lim_{x\to a} c_0] \\
&\overset{\text{Monomial, Thm 5.2}}{=} c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 \\
&= p(a).
\end{aligned}$$

The rational function result is simpler, still. If $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function, then $p(x)$ and $q(x)$ are polynomials. So for any point a in the domain of $r(x)$ (i.e., $q(a) \neq 0$),

$$\lim_{x\to a} r(x) = \lim_{x\to a}\left(\frac{p(x)}{q(x)}\right) \overset{\text{Quotient}}{=} \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} \overset{\text{Polynomial}}{=} \frac{p(a)}{q(a)} = r(a).$$

□

EXAMPLE 4.8. To see how these last results greatly simplify certain limit calculations, let's determine $\lim_{x\to 2} \dfrac{4x^2+2x}{3x+1}$.

SOLUTION. Since we have a rational function and the denominator is not 0 at $x = 2$, we see that

$$\lim_{x\to 2} \frac{4x^2+2x}{3x+1} \overset{\text{Rational}}{=} \frac{4(2)^2+2(2)}{3(2)+1} = \frac{12}{7}.$$

That was easy!

Several Cautions. Most of the limits we will encounter this term will not be so easy to determine. While we will use the properties we've developed and others below, most limits will start off in the indeterminate form $\dfrac{0}{0}$. Typically we will need to carry out some sort of algebraic manipulation to get the limit in a form where the basic properties apply. For example, while

$$\lim_{x\to 5} \frac{x^2-25}{x-5}$$

is a rational function, property 8 above does not apply to the calculation of the limit since 5 (the number x is approaching) is not in the domain of the function. Consequently, some algebraic manipulation (in this case factoring) is required.

$$\lim_{x\to 5} \frac{x^2-25}{x-5} = \lim_{x\to 5} \frac{(x-5)(x+5)}{x-5} = \lim_{x\to 5} x+5 \overset{\text{Poly}}{=} 10.$$

There are two additional things to notice. The first is *mathematical grammar*. We continue to use the limit symbol up until the actual numerical evaluation takes place. Writing something such as the following is simply wrong:

$$\xcancel{\lim_{x\to 5} \frac{x^2-25}{x-5} = \frac{(x-5)(x+5)}{x-5} = x+5 = 10.}$$

Among other things, the function $x+5$ is not the same as the constant 10.

An even worse calculation to write is

$$\xcancel{\lim_{x\to 5} \frac{x^2-25}{x-5} = \frac{0}{0} = 1}$$

or even

$$\xcancel{\lim_{x\to 5} \frac{x^2-25}{x-5} = \frac{0}{0} = \text{Undefined.}}$$

The expression $\frac{0}{0}$ is indeed not defined (and is certainly not equal to 1). However, the limit is indeterminate. Near (but not equal to) $x = 5$, the fraction is not yet $\frac{0}{0}$. You need to **do more work** to determine the limit. The work may involve factoring or other algebraic methods to simplify the expression so that we can more easily see what it is approaching.

Another thing to notice is that $\frac{x^2-25}{x-5}$ and $x+5$ are the *same* function as long as $x \neq 5$ where the first function is not defined but the second is. However, we are interested in a limit as $x \to 5$ so remember that this involves x being close to, *but not equal to*, 5. Consequently $\lim\limits_{x\to 5} \dfrac{x^2-25}{x-5}$ and $\lim\limits_{x\to 5} x+5$ are indeed the same!

4.3 *One-sided Limits*

We have now stated a number of properties for limits. All of these properties also hold for one-sided limits, as well, with a slight modification for fractional powers.

THEOREM 4.7 (One-sided Limit Properties). Limit properties 1 through 9 (the constant multiple, sum, difference, product, quotient, integer power, polynomial, and rational function rules) continue to hold for one-sided limits with the following modification for fractional powers

Assume that m and n are positive integers and that $\frac{n}{m}$ is reduced. Then

(*a*) $\lim\limits_{x\to a^+} [f(x)]^{n/m} = \left[\lim\limits_{x\to a^+} f(x)\right]^{n/m}$ provided that $f(x) \geq 0$ for x near a with $x > a$ if m is even.

(*b*) $\lim\limits_{x\to a^-} [f(x)]^{n/m} = \left[\lim\limits_{x\to a^-} f(x)\right]^{n/m}$ provided that $f(x) \geq 0$ for x near a with $x > a$ if m is even.

The next few examples illustrate the use of limit properties with piecewise functions.

EXAMPLE 4.9. Let $f(x) = \begin{cases} 3x^2+1, & \text{if } x < 2 \\ \sqrt{3x+9} & \text{if } x \geq 2 \end{cases}$. Determine the following limits if they exist.

(*a*) $\lim\limits_{x\to 2^-} f(x)$ (*b*) $\lim\limits_{x\to 2^+} f(x)$ (*c*) $\lim\limits_{x\to 2} f(x)$ (*d*) $\lim\limits_{x\to 0} f(x)$

SOLUTION. We must be careful to use the correct definition of f for each limit.

(a) As $x \to 2$ from the left, x is less than 2 so $f(x) = 3x^2 + 1$ there. Thus

$$\lim_{x\to 2^-} f(x) \overset{x \le 2}{=} \lim_{x\to 2^-} 3x^2 + 1 \overset{\text{Poly}}{=} 3(-2)^2 + 1 = 13.$$

(b) As $x \to 2$ from the right, x is greater than 2 so $f(x) = \sqrt{3x+9}$. Thus

$$\lim_{x\to 2^+} f(x) \overset{x \ge 2}{=} \lim_{x\to 2^+} \sqrt{3x+9} \overset{\text{Root}}{=} \sqrt{15}.$$

(c) To determine $\lim\limits_{x\to 2} f(x)$ we compare the one sided limits. Since $\lim\limits_{x\to 2^+} f(x) \neq \lim\limits_{x\to 2^-} f(x)$, we conclude that $\lim\limits_{x\to 2} f(x)$ DNE.

(d) To determine $\lim\limits_{x\to 0} f(x)$ we see that the values of x near 0 are less than 2. So $f(x) = 3x^2 + 1$ there. So

$$\lim_{x\to 0} f(x) \overset{x \le 2}{=} \lim_{x\to 0} 3x^2 + 1 \overset{\text{Poly}}{=} 1.$$

We don't need to use the other definition for f since it does not apply to values of x near 0.

EXAMPLE 4.10. Let $f(x) = \begin{cases} 3x-1, & \text{if } x \le 1 \\ x^2+1, & \text{if } 1 < x \le 5 \\ \frac{x}{x+1} & \text{if } x > 5 \end{cases}$. Determine the following limits if they exist.

(a) $\lim\limits_{x\to 1^-} f(x)$ (b) $\lim\limits_{x\to 1^+} f(x)$ (c) $\lim\limits_{x\to 1} f(x)$

(d) $\lim\limits_{x\to 5^-} f(x)$ (e) $\lim\limits_{x\to 5^+} f(x)$ (f) $\lim\limits_{x\to 5} f(x)$

SOLUTION. We must be careful to use the correct definition of f for each limit. Note how we choose the function!

(a) $\lim\limits_{x\to 1^-} f(x) \overset{x \le 1}{=} \lim\limits_{x\to 1^-} 3x - 1 \overset{\text{Poly}}{=} 2.$

(b) $\lim\limits_{x\to 1^+} f(x) \overset{1 < x \le 5}{=} \lim\limits_{x\to 1^+} x^2 + 1 \overset{\text{Poly}}{=} 2.$

(c) Since $\lim\limits_{x\to 1^+} f(x) = 2 = \lim\limits_{x\to 1^-} f(x)$, we conclude that $\lim\limits_{x\to 1} f(x) = 2$.

(d) $\lim\limits_{x\to 5^-} f(x) \overset{1 < x \le 5}{=} \lim\limits_{x\to 5^-} x^2 + 1 \overset{\text{Poly}}{=} 26.$

(e) $\lim\limits_{x\to 5^+} f(x) \overset{x \ge 5}{=} \lim\limits_{x\to 5^+} \dfrac{x}{x+1} \overset{\text{Rat'l}}{=} \dfrac{5}{6}.$

(f) Since $\lim\limits_{x\to 5^+} f(x) \neq \lim\limits_{x\to 5^-} f(x)$, we conclude that $\lim\limits_{x\to 5} f(x)$ DNE.

4.4 *Most Limits Are Not Simple*

Let's return to the original motivation for calculating limits. We were interested in finding the 'slope' of a curve and this led to looking at limits that have the form

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Assuming that f is continuous, this limit cannot be evaluated by any of the basic limit properties since the denominator is approaching 0. More specifically, as $x \to a$, this difference quotient has the **indeterminate form** $\frac{0}{0}$. To evaluate this limit we must do more work. Let's look at an

EXAMPLE 4.11. Let $f(x) = x^2 - 3x + 1$. Determine the slope of this curve right at $x = 4$.

SOLUTION. To find the slope of a curve we must evaluate the difference quotient

$$\lim_{x\to 4} \frac{f(x)-f(4)}{x-4} = \lim_{x\to 4} \frac{(x^2-3x+1)-5}{x-4}.$$

Though this is a rational function, the limit properties do not apply since the denominator is 0 at 4, and so is the numerator (check it!). Instead, we must 'do more work.'

$$\lim_{x\to 4} \frac{f(x)-f(4)}{x-4} = \lim_{x\to 4} \frac{x^2-3x-4}{x-4} = \lim_{x\to 4} \frac{(x-4)(x+1)}{x-4} = \lim_{x\to 4} x+1 \overset{\text{Poly}}{=} 5.$$

Only at the very last step were we able to use a limit property.

The Indeterminate Form $\frac{0}{0}$

Many of the most important limits we will see in the course have the indeterminate form $\frac{0}{0}$ as in the previous example. To evaluate such limits, if they exist, requires 'more work' typically of the following type.

- factoring
- using conjugates[2]
- simplifying
- making use of known limits

Let's look at some examples of each.

[2] Recall that if $a > 0$, then $\sqrt{a}+b$ and $\sqrt{a}-b$ are called **conjugates**. Notice that

$$(\sqrt{a}+b)(\sqrt{a}-b) = a-b^2.$$

There is no middle term.

EXAMPLE 4.12 (Factoring). Factoring is one of the most critical tools in evaluating the sorts of limits that arise in elementary calculus. Evaluate $\lim\limits_{x\to 2} \dfrac{2x^2-6x+4}{x^2+2x-8}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Factoring is the key.

$$\lim_{x\to 2} \frac{2x^2-6x+4^{\nearrow 0}}{x^2+2x-8_{\searrow 0}} = \lim_{x\to 2} \frac{2(x-1)(x-2)}{(x+4)(x-2)} = \lim_{x\to 2} \frac{2(x-1)}{x+4} \overset{\text{Rational}}{=} \frac{2}{6} = \frac{1}{3}.$$

Only at the very last step were we able to use a limit property.

EXAMPLE 4.13 (Factoring). Evaluate $\lim\limits_{x\to -2} \dfrac{x^2+8x+12}{x^3+2x^2}$.

SOLUTION. This limit has the indeterminate form $\frac{0}{0}$. Factoring is the key.

$$\lim_{x\to -2} \frac{x^2+8x+12^{\nearrow 0}}{x^3+2x^2_{\searrow 0}} = \lim_{x\to -2} \frac{(x+6)(x+2)}{x^2(x+2)} = \lim_{x\to -2} \frac{(x+6)}{x^2} \overset{\text{Rat'l}}{=} \frac{4}{4} = 1.$$

Only at the very last step were we able to use a limit property.

EXAMPLE 4.14. (Conjugates) Evaluate $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{2x-8)}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Let's see how conjugates help.

$$\begin{aligned}
\lim_{x\to 4} \frac{\sqrt{x}-2^{\nearrow 0}}{2(x-4)_{\searrow 0}} = \lim_{x\to 4} \frac{\sqrt{x}-2}{2x-8} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} &= \lim_{x\to 4} \frac{x-4}{(2x-8)(\sqrt{x}+2)} \\
&= \lim_{x\to 4} \frac{x-4}{(2(x-4)(\sqrt{x}+2)} \\
&= \lim_{x\to 4} \frac{1}{2(\sqrt{x}+2)} \overset{\text{Root}}{=} \frac{1}{2\sqrt{4}+2} = \frac{1}{8}.
\end{aligned}$$

EXAMPLE 4.15. (Conjugates) Here's another: Evaluate $\lim\limits_{x\to 2} \dfrac{\sqrt{x+4}-\sqrt{6}}{x-2}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use conjugates again.

$$\begin{aligned}
\lim_{x\to 2}\frac{\sqrt{x+4}-\sqrt{6}^{\nearrow 0}}{x-2_{\searrow 0}} &= \lim_{x\to 2}\frac{\sqrt{x+4}-\sqrt{6}}{x-2}\cdot\frac{\sqrt{x+4}+\sqrt{6}}{\sqrt{x+4}+\sqrt{6}}\\
&= \lim_{x\to 2}\frac{(x+4)-6}{(x-2)(\sqrt{x+4}+\sqrt{6})}\\
&= \lim_{x\to 2}\frac{x-2}{(x-2)(\sqrt{x+4}+\sqrt{6})}\\
&= \lim_{x\to 2}\frac{1}{\sqrt{x+4}+\sqrt{6}}\\
&\overset{\text{Root}}{=} \frac{1}{2\sqrt{6}}.
\end{aligned}$$

EXAMPLE 4.16. (Conjugates) Evaluate $\lim_{x\to 1}\frac{x^2-1}{\sqrt{x+3}-2}$.

SOLUTION. This limit has the indeterminate form $\frac{0}{0}$.

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2-1^{\nearrow 0}}{\sqrt{x+3}-2_{\searrow 0}} &= \lim_{x\to 1}\frac{x^2-1}{\sqrt{x+3}-2}\cdot\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\\
&= \lim_{x\to 1}\frac{(x^2-1)(\sqrt{x+3}+2)}{(x+3)-4}\\
&= \lim_{x\to 1}\frac{(x-1)(x+1)(\sqrt{x+3}+2)}{x-1}\\
&= \lim_{x\to 1}(x+1)(\sqrt{x+3}+2)\\
&\overset{\text{Prod, Root}}{=} 2(2+2)=8.
\end{aligned}$$

EXAMPLE 4.17 (Simplification)**.** Sometimes limits, like this next one, involve compound fractions. One method of attack is to carefully simplify them. Evaluate $\lim_{x\to 2}\frac{\frac{2}{x+1}-\frac{2}{x^2-1}}{x-2}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\begin{aligned}
\lim_{x\to 2}\frac{\frac{2}{x+1}-\frac{2}{x^2-1}^{\nearrow 0}}{x-2_{\searrow 0}} &= \lim_{x\to 2}\frac{\frac{2(x-1)-2}{(x+1)(x-1)}}{x-2}\\
&= \lim_{x\to 2}\frac{2x-4^{\nearrow 0}}{(x+1)(x-1)(x-2)_{\searrow 0}}\\
&= \lim_{x\to 2}\frac{2}{(x+1)(x-1)} \overset{\text{Rational}}{=} \frac{2}{3}.
\end{aligned}$$

EXAMPLE 4.18 (Simplification)**.** Evaluate $\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}^{\nearrow 0}}{x-1_{\searrow 0}} = \lim_{x\to 1}\frac{\frac{2-(x+1)}{2(x+1}}{x-1} = \lim_{x\to 2}\frac{1-x^{\nearrow 0}}{2(x+1)(x-1)_{\searrow 0}} = \lim_{x\to 2}\frac{-1}{2(x+1)} \overset{\text{Rat'l}}{=} \frac{1}{2}.$$

EXAMPLE 4.19 (Simplification)**.** Evaluate $\lim_{h\to 0}\frac{\frac{2}{x+h}-\frac{2}{x}}{h}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$.

$$\lim_{h\to 0}\frac{\frac{2}{x+h}-\frac{2}{x}^{\nearrow 0}}{h_{\searrow 0}} = \lim_{h\to 0}\frac{\frac{2x-2(x+h)}{(x+h)(x)}}{h} = \lim_{h\to 0}\frac{-2h}{(x+h)(x)(h)} = \lim_{h\to 0}\frac{-2}{(x+h)(x)} \overset{\text{Rat'l}}{=} -\frac{2}{x^2}.$$

EXAMPLE 4.20 (Simplification)**.** Evaluate $\lim_{x\to 1} \dfrac{\frac{4}{x^2+7} - \frac{1}{2}}{x-1}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\lim_{x\to 1} \frac{\frac{4}{x^2+7} - \frac{1}{2}}{x-1} = \lim_{x\to 1} \frac{\frac{8-(x^2+7)}{2(x^2+7)}}{x-1} = \lim_{x\to 1} \frac{\frac{1-x^2}{2(x^2+7)}}{x-1} = \lim_{x\to 1} \frac{(1-x)(1+x)}{2(x^2+7)(x-1)}$$
$$= \lim_{x\to 1} \frac{-(1+x)}{2(x^2+7)}$$
$$\overset{\text{Rat'l}}{=} \frac{-2}{16} = -\frac{1}{8}.$$

4.5 Practice Problems

EXAMPLE 4.21 (Simplification)**.** Evaluate $\lim_{x\to 4} \dfrac{x^2-6x+8}{x-4}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use factoring to simplify this rational function.

$$\lim_{x\to 4} \frac{x^2-6x+8}{x-4} = \lim_{x\to 4} \frac{(x-4)(x-2)}{x-4} = \lim_{x\to 4} x-2 \overset{\text{Linear}}{=} 2.$$

EXAMPLE 4.22 (Simplification)**.** Evaluate $\lim_{x\to -2} \dfrac{x+2}{4-x^2}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use factoring to simplify this rational function.

$$\lim_{x\to -2} \frac{x+2}{4-x^2} = \lim_{x\to -2} \frac{x+2}{(2-x)(2+x)} = \lim_{x\to -2} \frac{1}{2-x} \overset{\text{Rat'l}}{=} \frac{1}{4}.$$

EXAMPLE 4.23 (Simplification)**.** Evaluate $\lim_{x\to 5} \dfrac{x^2-3x-10}{x^2-25}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use factoring to simplify this rational function.

$$\lim_{x\to 5} \frac{x^2-3x-10}{x^2-25} = \lim_{x\to 5} \frac{(x+2)(x-5)}{(x-5)(x+5)} = \lim_{x\to 5} \frac{x+2}{x+5} \overset{\text{Rat'l}}{=} \frac{7}{10}.$$

EXAMPLE 4.24 (Simplification)**.** Evaluate $\lim_{x\to -1} \dfrac{x^3-x}{x^2-5x-6}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use factoring to simplify this rational function.

$$\lim_{x\to -1} \frac{x^3-x}{x^2-5x-6} = \lim_{x\to -1} \frac{x(x^2-1)}{(x+1)(x-6)} = \lim_{x\to -1} \frac{x(x-1)(x+1)}{(x+1)(x-6)} = \lim_{x\to -1} \frac{x(x-1)}{x-6}$$
$$\overset{\text{Rat'l}}{=} -\frac{2}{7}.$$

EXAMPLE 4.25 (Simplification)**.** Evaluate $\lim_{x\to 0} \dfrac{\frac{3}{2x+1} - 3}{x}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\lim_{x\to 0} \frac{\frac{3}{2x+1} - 3}{x} = \lim_{x\to 0} \frac{\frac{3-3(2x+1)}{2x+1}}{x} = \lim_{x\to 0} \frac{\frac{3-6x+3}{2x+1}}{x} = \lim_{x\to 0} \frac{-6x}{x(2x+1)} = \lim_{x\to 2} \frac{-6}{2x+1} \overset{\text{Rat'l}}{=} -6.$$

EXAMPLE 4.26 (Simplification)**.** Evaluate $\lim_{x\to 2} \dfrac{\frac{1}{x^2}-\frac{1}{4}}{2-x}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\lim_{x\to 2} \frac{\frac{1}{x^2}-\frac{1}{4}}{2-x} = \lim_{x\to 2} \frac{\frac{4-x^2}{4x^2}}{2-x} = \lim_{x\to 2} \frac{4-x^2}{(4x^2)(2-x)} = \lim_{x\to 2} \frac{(2-x)(2+x)}{(4x^2)(2-x)} = \lim_{x\to 2} \frac{2+x}{4x^2} \stackrel{\text{Rat'l}}{=} \frac{4}{16} = \frac{1}{4}.$$

EXAMPLE 4.27 (Simplification)**.** Evaluate $\lim_{x\to 1} \dfrac{\frac{1}{x^2+1}-\frac{1}{2}}{x-1}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\begin{aligned}\lim_{x\to 1} \frac{\frac{1}{x^2+1}-\frac{1}{2}}{x-1} = \lim_{x\to 1} \frac{\frac{2-x^2-1}{2(x^2+1)}}{x-1} = \lim_{x\to 1} \frac{\frac{1-x^2}{2(x^2+1)}}{x-1} &= \lim_{x\to 1} \frac{1-x^2}{2(x^2+1)(x-1)} \\ &= \lim_{x\to 1} \frac{(1-x)(1+x)}{2(x^2+1)(x-1)} \\ &= \lim_{x\to 1} \frac{-(1+x)}{2(x^2+1)} \\ &\stackrel{\text{Rat'l}}{=} \frac{-2}{4} = -\frac{1}{2}.\end{aligned}$$

EXAMPLE 4.28 (Simplification)**.** Evaluate $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use conjugates to simplify.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} \stackrel{\text{Root}}{=} \frac{1}{2}.$$

EXAMPLE 4.29 (Simplification)**.** Evaluate $\lim_{x\to 3} \dfrac{x-3}{\sqrt{x+1}-2}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use conjugates to simplify.

$$\begin{aligned}\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2} = \lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} &= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+1}+2)}{x+1-4} \\ &= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+1}+2)}{x-3} \\ &= \lim_{x\to 3} \sqrt{x+1}+2 \stackrel{\text{Root}}{=} 4.\end{aligned}$$

EXAMPLE 4.30 (Simplification)**.** Evaluate $\lim_{x\to 0} \dfrac{\sqrt{4-x}-2}{x^2-x}$.

SOLUTION. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use conjugates to simplify.

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{4-x}-2}{x^2-x} = \lim_{x\to 0} \frac{\sqrt{4-x}-2}{x^2-x} \cdot \frac{\sqrt{4-x}+2}{\sqrt{4-x}+2} &= \lim_{x\to 0} \frac{(4-x)-4}{(x^2-x)(\sqrt{4-x}+2)} \\ &= \lim_{x\to 0} \frac{-x}{(x^2-x)(\sqrt{4-x}+2)} \\ &= \lim_{x\to 0} \frac{-x}{x(x-1)(\sqrt{4-x}+2)} \\ &= \lim_{x\to 0} \frac{-1}{(x-1)(\sqrt{4-x}+2)} \\ &\stackrel{\text{Root, Prod}}{=} \frac{-1}{-4} = \frac{1}{4}.\end{aligned}$$