

More on Continuity

Quick Review

Continuity Checklist. A function f is continuous at a if the following three conditions hold:

1. $f(a)$ is defined (i.e., a is in the domain of f).
2. $\lim_{x \rightarrow a} f(x)$ exists.
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

Remember:

- A polynomial is continuous everywhere, i.e. for all x .
- A rational functions $r(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials is continuous at all points in its domain, i.e., where $q(x) \neq 0$.

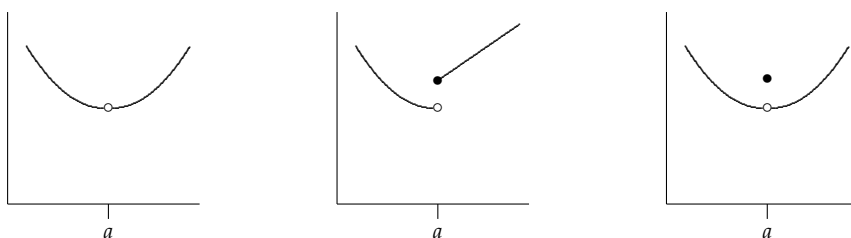


Figure 9.4: Three of the ways in which a function f may fail to be continuous because $\lim_{x \rightarrow a} f(x) \neq f(a)$.

12. $f(a)$ may not be defined;
13. $\lim_{x \rightarrow a} f(x)$ may not exist;
14. $\lim_{x \rightarrow a} f(x)$ and $f(a)$ may both exist but not be equal.

EXAMPLE 9.7. Determine the points at which $f(x) = \frac{\frac{2}{x} - \frac{2}{5}}{x - 5}$ is discontinuous. At which points does f have VA's? Removable discontinuities?

SOLUTION. Since $f(x)$ is rational it is continuous at all points in its domain. We can see immediately that f is *not defined* at $x = 5$ and $x = 0$, where there would be division by 0, so f is not continuous at these two points. Let's simplify the expression for f before taking the appropriate limits.

$$f(x) = \frac{\frac{2}{x} - \frac{2}{5}}{x - 5} = \frac{\frac{10-2x}{5x}}{x - 5} = \frac{10 - 2x}{5x(x - 5)}.$$

At $x = 0$:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{10 - 2x}{5x(x - 5)} = \lim_{x \rightarrow 0^-} \frac{-2(x - 5)}{5x(x - 5)} = \lim_{x \rightarrow 0^-} \underbrace{\frac{-2}{5x}}_{5 \cdot 0^- = 0^-} = \infty.$$

This is enough to conclude that f has a VA at 0.

At $x = 5$: Having seen the factorization of f , we know that we can calculate a two-sided limit at 5.

$$\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} \frac{-2(x - 5)}{5x(x - 5)} = \lim_{x \rightarrow 5} \frac{-2}{5x} \stackrel{\text{Rat'l}}{=} \frac{-2}{25}.$$

Since $\lim_{x \rightarrow 5} f(x)$ exists but $f(5)$ is not defined, then f has a removable discontinuity at 5.

Basic Continuity Properties

One of the most important facts about continuity is that it is preserved under the standard mathematical operations on functions.

THEOREM 9.2. (Some Types of Continuous Functions) Assume that f and g are both continuous at $x = a$ and that c is a constant. Then the following new functions are continuous at $x = a$.

- (a) The sum and differences: $f + g$ and $f - g$
- (b) Constant multiples: cf
- (c) Products: fg
- (d) Quotients: $\frac{f}{g}$, provided $g(a) \neq 0$
- (e) Powers: $[f(x)]^n$, where n is a positive integer.

Proof. The proofs of all of these follow from the corresponding basic limit properties. Let's prove (c). Since f and g are continuous, by Definition 8.1 this means $\lim_{x \rightarrow a} f(x) = f(a)$ and $\lim_{x \rightarrow a} g(x) = g(a)$. But then by the product limit property,

$$\lim_{x \rightarrow a} [f(x)g(x)] \stackrel{\text{Product}}{=} \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) \stackrel{f, g \text{ cont}}{=} f(a)g(a)$$

so fg is continuous at a . The other parts can be proven in a similar fashion. $\square$

YOU TRY IT 9.5. Try proving part (e). Which limit property must you use?

YOU TRY IT 9.6 (Think it through). Earlier we the basic limit properties we were able to show earlier that polynomials were continuous. INstead we could have waited until now to show polynomials are continuous using the continuity properties in Theorem 9.2. We know that $f(x) = x$ is a continuous function. Do you see how you prove that x^2 is continuous everywhere directly from Theorem 9.2? What parts of Theorem 9.2 would you need to use to prove that $2x^2 + 3x$ is continuous everywhere? How about a general polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$? Once you know that polynomials are continuous, which property in Theorem 9.2 tells you that rational functions are continuous on their domains?

Composition and Continuity

So far we have not mentioned the operation of composition. This is the most complex of the basic function operations, nonetheless it preserves continuity under the appropriate circumstances. Functions, like numbers, may be combined using sums, differences, products, or quotients as we have seen in our limit properties. The process called *composition* also produces new functions and this process is unique to functions and is not possible to carry out on numbers. Composition of functions will play an important role in our development of the calculus. Recall how composition works.

DEFINITION (Composite Functions). Given two functions f and g , the **composite function** $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. It is evaluated in two steps: $y = f(u)$ where $u = g(x)$. The domain of $f \circ g$ consists of all x in the domain of g such that $u = g(x)$ is in the domain of f .

EXAMPLE 9.8. Let $f(x) = 3x^2 - x$, $g(x) = x^2 - 1$, $h(x) = \sqrt{x}$, and $j(x) = \frac{1}{x}$. Simplify the following expressions. Determine their domains.

- (a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$ (c) $(h \circ f)(x)$
- (d) $(g \circ g)(x)$ (e) $(j \circ j)(x)$ (f) $(f \circ j)(x)$

SOLUTION. We use the definition of composite function.

- (a) $(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = 3(x^2 - 1)^2 - (x^2 - 1) = 3x^4 - 6x^2 + 3 - x^2 + 1 = 3x^4 - 7x^2 + 4$. The domain is all x since both functions are polynomials.
- (b) $(g \circ f)(x) = g(f(x)) = g(3x^2 - x) = (3x^2 - x)^2 - 1 = 9x^4 - 6x^3 + x^2 - 1$. The domain is all x as above. This illustrates the fact that $(f \circ g)(x)$ and $(g \circ f)(x)$ are quite different functions.
- (c) $(h \circ f)(x) = h(f(x)) = h(3x^2 - x) = \sqrt{3x^2 - x}$. This time the domain of f is all x but the domain of $h \circ f$ requires $f(x) = 3x^2 - x = x(3x - 1)$ to be at least 0. So we need $x \geq \frac{1}{3}$ or $x \leq 0$.
- (d) $(g \circ g)(x) = g(g(x)) = g(x^2 - 1) = (x^2 - 1)^2 - 1 = x^4 - 2x^2$. The domain is all x since g is a polynomial.
- (e) $(j \circ j)(x) = j(j(x)) = j(\frac{1}{x}) = \frac{1}{\frac{1}{x}} = x$. Careful: The domain is all $x \neq 0$ since x must be in the domain of j to start with.
- (f) $(f \circ j)(x) = f(j(x)) = f(\frac{1}{x}) = 3(\frac{1}{x})^2 - \frac{1}{x} = \frac{3}{x^2} - \frac{1}{x}$. The domain is all $x \neq 0$.

THEOREM 9.3. (Continuity and Composition) If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a . That is, $\lim_{x \rightarrow a} f(g(x)) = f(g(a))$.

The proof of this theorem is harder than the previous results—take Math 331. The theorem can also be thought of as a *switch in the order of the limit and composition operations*. Since g is continuous at a , this means that $\lim_{x \rightarrow a} g(x) = g(a)$. So the theorem says

$$\lim_{x \rightarrow a} f(g(x)) \stackrel{\text{Cont}}{=} f \left(\overbrace{g(a)}^{\lim_{x \rightarrow a} g(x)} \right) = f \left(\lim_{x \rightarrow a} g(x) \right). \quad (9.1)$$

In other words, we can switch the order of the limit and composition operations.

EXAMPLE 9.9 (Composition). Determine $\lim_{x \rightarrow 1} (x^4 - 3)^6$.

SOLUTION. OK, we could multiply out the polynomial to the sixth power and take the limit since it will still be a polynomial. (Good luck!) Or we can think about $(x^4 - 3)^6$ as a composition: Let $g(x) = x^4 - 3$ and $f(x) = x^6$. Then

$$(x^4 - 3)^6 = f(g(x)).$$

Since both f and g are polynomials, both are continuous everywhere, so Theorem 9.4 applies. So switching the order of operations

$$\lim_{x \rightarrow 1} (x^4 - 3)^6 \stackrel{\text{Theorem 9.4}}{=} \left(\lim_{x \rightarrow 1} (x^4 - 3) \right)^6 \stackrel{\text{Poly}}{=} (1 - 3)^6 = 64.$$

Continuity on Intervals

As you might expect, a function is *continuous on an interval* if it is continuous at every point in the interval. Sounds simple, right? But what about at the endpoints when the interval is closed? Take a look at graph of $f(x)$ in Figure 9.5. What are the largest intervals on which f is continuous? We need to define the notion of one-sided continuity to deal with this question.

DEFINITION 9.3. (Left and Right Continuity) A function f is **left-continuous** at a if $\lim_{x \rightarrow a^-} f(x) = f(a)$ and f is **right-continuous** at a if $\lim_{x \rightarrow a^+} f(x) = f(a)$.

It is worth pointing out the obvious. Since the two-sided limit exists at a if and only if the two one-sided limits exist and are equal, it follows that

THEOREM 9.4. f is continuous at a if and only if f is both left and right continuous at a .

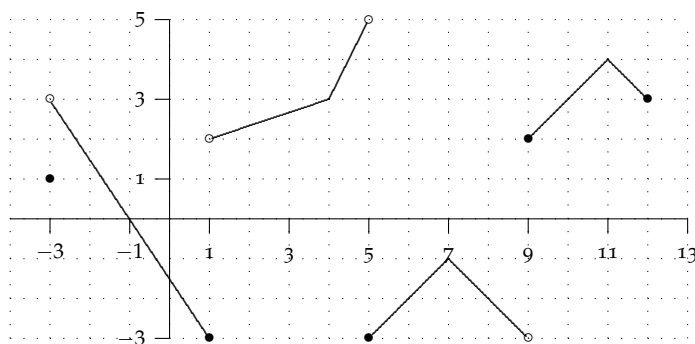


Figure 9.5: f is left-continuous at $x = 1$ and right-continuous at $x = 9$.

Let's get back to intervals. Now we combine this definition with the definition of continuity at a point (Definition 8.1) to define continuity on an interval.

DEFINITION 9.4. (Continuity on an Interval) A function f is **continuous on an interval** I if f is continuous at all the points of I . If I contains its endpoints, then continuity on I means left- or right-continuous the right or left endpoints, respectively.

YOU TRY IT 9.7. Use Figure 9.5 to answer these questions. Notice that f is continuous on the interval $(-3, 1]$ since it is continuous from the left at 1 but not from the right at -3 . Further f is not continuous a larger interval containing this one since f is not continuous (from both sides) at either -3 or 1.

- Find the largest interval contain $x = 2$ on which f is continuous. Explain.
- Find the largest interval contain $x = 5$ on which f is continuous. Explain.
- Find the largest interval contain $x = 7$ on which f is continuous. Explain.
- Find the largest interval contain $x = 9$ on which f is continuous. Explain.

EXAMPLE 9.10 (Intervals of Continuity). Determine the largest intervals of continuity for

$$f(x) = \begin{cases} \frac{x^2+1}{x-1} & \text{if } x > 2 \\ -2 & \text{if } x = 2 \\ \frac{x^2-2x}{x^2-5x+6} & \text{if } x < 2 \end{cases}$$

SOLUTION. The piecewise function consists of two rational functions and a value at a single point. On $(2, \infty)$, $f(x) = \frac{x^2+1}{x-1}$ is continuous by Theorem 8.1 since the denominator is never 0. So f is continuous at least on $(2, \infty)$. Is f right-continuous at 2? Well, we know that $f(2) = -2$. And

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{x^2+1}{x-1} = \frac{5}{1} = 5 \neq -2.$$

So f is not right continuous at 2. So one of the intervals of continuity is $(2, \infty)$.

On the interval $(-\infty, 2)$, $f(x) = \frac{x^2-2x}{x^2-5x+6} = \frac{x(x-2)}{(x-2)(x-3)}$ is rational and continuous by Theorem 8.1 since the denominator is never 0 there. Is f left-continuous at -2 ?

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{x^2-2x}{x^2-5x+6} = \lim_{x \rightarrow 2^-} \frac{x}{x-3} = \frac{2}{-1} = -2 = f(2).$$

So f is left-continuous at 2. Therefore, f is continuous on $(-\infty, 2]$.

Note, however that f is not continuous at 2 since the two one-sided limits are different there.

EXAMPLE 9.11 (Intervals of Continuity). Determine the largest intervals of continuity for

$$f(x) = \begin{cases} \frac{x+1}{x} & \text{if } x > 0 \\ x^2 - x - 1 & \text{if } x \leq 0 \end{cases}$$

Answers to **YOU TRY IT 9.7**: (a) $(1, 5)$ since f is not left-continuous at 5 nor right-continuous at 1. (b) $[5, 9)$. (c) $[5, 9)$. (d) $[9, 11]$.

SOLUTION. The piecewise function consists of a rational function and a polynomial. On $(0, \infty)$, $f(x) = \frac{x+1}{x}$ is continuous by Theorem 8.1 since the denominator is never 0. So f is continuous at least on $(0, \infty)$. Is f right-continuous at 0? Well, we know that $f(0) = 0^2 - 0 = 0$. And

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\overbrace{x+1}^1}{\underbrace{x}_{0^+}} = +\infty$$

So f is not right continuous at 0. So one of the intervals of continuity is $(0, \infty)$.

On the interval $(-\infty, 0)$, $f(x) = x^2 - x - 1$ is polynomial and is continuous by Theorem 8.1. Is f left-continuous at 0?

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 - x - 1 \stackrel{\text{Poly}}{=} -1 = f(0).$$

So f is left-continuous at 0. Therefore, f is continuous on $(-\infty, 0]$. Note, however that f is not continuous at 0 since the two one-sided limits are different there.

EXAMPLE 9.12 (Intervals of Continuity). Determine the largest intervals of continuity for

$$f(x) = \begin{cases} x+2 & \text{if } x \geq 3 \\ x^2 - x - 1 & \text{if } x < 3 \end{cases}.$$

SOLUTION. The piecewise function consists of two polynomials. On $(3, \infty)$, $f(x) = x + 2$ is a continuous polynomial. Is f right-continuous at 3? Well, we know that $f(3) = 3 + 2 = 5$. And

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} x + 2 = 5$$

So f is right continuous at 3. So one of the intervals of continuity is $[3, \infty)$.

On the interval $(-\infty, 3)$, $f(x) = x^2 - x - 1$ is polynomial and is continuous by Theorem 8.1. Is f left-continuous at 3?

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} x^2 - x - 1 \stackrel{\text{Poly}}{=} 5 = f(3).$$

So f is left-continuous at 3. Therefore, f is continuous on $(-\infty, 0]$. In fact, because the two one-sided limits at 3 are equal,

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = 5,$$

we have $\lim_{x \rightarrow 3} f(x) = 5 = f(3)$. So f is continuous at 3 (and everywhere else since it is a polynomial if $x \neq 3$). So $f(x)$ is actually continuous for all x , i.e., on $(-\infty, \infty)$.