Still More on Continuity

Continuity on Intervals

As you might expect, a function is *continuous on an interval* if it is continuous at every point in the interval. Sounds simple, right? But what about at the endpoints when the interval is closed? Take a look at graph of $f(x)$ in Figure 10.5. What are the largest intervals on which f is continuous? We need to define the notion of one-sided continuity to deal with this question.
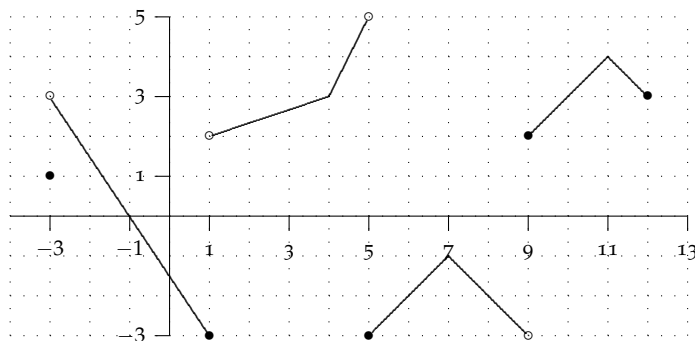
DEFINITION 10.3. (Left and Right Continuity) A function f is **left-continuous** at a if $\lim_{x \to a^-} f(x) = f(x)$ and f is **right-continuous** at a if $\lim_{x \to a^+} f(x) = f(x)$.

It is worth pointing out the obvious. Since the two-sided limit exists at a if and only if the two one-sided limits exist and are equal, it follows that

THEOREM 10.4. f is continuous at a if and only if f is both left and right continuous at a.

Let's get back to intervals Now we combine this definition with the definition of continuity at a point (Definition 8.1) to define continuity on an interval.

DEFINITION 10.4. (Continuity on an Interval) A function f is **continuous on an interval** I if f is continuous at all the points of I. If I contains its endpoints, then continuity on I means left- or right-continuous the right or left endpoints, respectively.

YOU TRY IT 10.7. Use Figure 10.5 to answer these questions. Notice that f is continuous on the interval $(-3, 1]$ since it is continuous from the left at 1 but not from the right at -3. Further f is not continuous a larger interval containing this one since f is not continuous (from both sides) at either -3 or 1.

   (*a*) Find the largest interval contain $x = 2$ on which f is continuous. Explain.

   (*b*) Find the largest interval contain $x = 5$ on which f is continuous. Explain.

   (*c*) Find the largest interval contain $x = 7$ on which f is continuous. Explain.

   (*d*) Find the largest interval contain $x = 9$ on which f is continuous. Explain.

Answers to YOU TRY IT 10.7 : (a) $(1, 5)$ since f is not left-continuous at 5 nor right-continuous at 1. (b) $[5, 9)$. (c) $[5, 9)$. (d) $[9, 11]$.

EXAMPLE 10.10 (Intervals of Continuity). Let's do a complete analysis of the function Determine the largest intervals of continuity for

$$f(x) = \begin{cases} \frac{x^2+1}{x-1} & \text{if } x > 2 \\ -2 & \text{if } x = 2 \\ \frac{x^2-2x}{x^2-5x+6} & \text{if } x < 2 \end{cases}.$$

The piecewise function consists of two rational functions and a value at a single point. On $(2, \infty)$, $f(x) = \frac{x^2+1}{x-1}$ is continuous by Theorem 8.1 since the denominator is never 0. So f is continuous at least on $(2, \infty)$. On the interval $(-\infty, 2)$, $f(x) = \frac{x^2-2x}{x^2-5x+6} = \frac{x(x-2)}{(x-2)(x-3)}$ is rational and continuous by Theorem 8.1 since the denominator is never 0 there. The only problem is a $x = 2$.

Questions: Is $f(x)$ left or right continuous, or just plain continuous at $x = 2$? State the intervals of continuity for f using interval notation. Does it have an RD at $x = 2$? Does f have a VA at $x = 2$? State the intervals of continuity for f using interval notation. [Make sure you justify your answers as we have done below using limits.]

SOLUTION. Is f right-continuous at 2? Well, we know that $f(2) = -2$. And

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x^2+1}{x-1} = \frac{5}{1} = 5 \neq -2.$$

So f is not right continuous at 2. So one of the intervals of continuity is $(2, \infty)$.
    Is f left-continuous at -2?

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2-2x}{x^2-5x+6} = \lim_{x \to 2^-} \frac{x}{x-3} = \frac{2}{-1} = -2 = f(2).$$

So f is left-continuous at 2. Therefore, f is continuous on $(-\infty, 2]$.
    However that f is not continuous at 2 since f is not continuous from **both** the left and the right.
    In the end, the intervals of continuity are $(-\infty, 2] \cup (2, \infty)$.
    Is there an RD at $x = 2$? No. Notice that $\lim_{x \to 2} f(x)$ DNE because the left and right limits are different. (Remember: To have an RD, we need $\lim_{x \to 2} f(x)$ to exist and be different from $f(2)$.)
    Is there a VA at $x = 2$? No. Neither the left or the right limits are infinite. (Remember: To have a VA, we need $\lim_{x \to 2^-} f(x) = \pm\infty$ or $\lim_{x \to 2^+} f(x) = \pm\infty$.)

EXAMPLE 10.11 (Intervals of Continuity). Let's do a complete analysis of the function

$$f(x) = \begin{cases} \frac{x+1}{x} & \text{if } x > 0 \\ x^2 - x - 1 & \text{if } x \leq 0 \end{cases}.$$

The piecewise function consists of a rational function and a polynomial. On $(0, \infty)$, $f(x) = \frac{x+1}{x}$ is continuous by Theorem 8.1 since the denominator is never 0. So f is continuous at least on $(0, \infty)$. On the interval $(-\infty, 0)$, $f(x) = x^2 - x - 1$ is polynomial and is continuous by Theorem 8.1. The only problem is a $x = 0$.
    Questions: Is $f(x)$ left or right continuous, or just plain continuous at $x = 0$? State the intervals of continuity for f using interval notation. Does it have an RD at $x = 0$? Does f have a VA at $x = 0$?

SOLUTION. Is f right-continuous at 0? Well, we know that $f(0) = 0^2 - 0 = 0$. And

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\overbrace{x+1}^{1}}{\underbrace{x}_{0^+}} = +\infty$$

So f is not right continuous at 0. So one of the intervals of continuity is $(0, \infty)$.

On the interval $(-\infty, 0)$, $f(x) = x^2 - x - 1$ is polynomial and is continuous by Theorem 8.1. Is f left-continuous at 0?

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 - x - 1 \overset{\text{Poly}}{=} -1 = f(0).$$

So f is left-continuous at 0. Therefore, f is continuous on $(-\infty, 0]$.

Note, however that f is not continuous at 0 since the two one-sided limits are different there.

In the end, the intervals of continuity are $(-\infty, 0,] \cup (0, \infty)$.

Is there an RD at $x = 0$? No. Notice that $\lim_{x \to 0} f(x)$ DNE because the left and right limits are different. (Remember: To have an RD, we need $\lim_{x \to 0} f(x)$ to exist and be different from $f(2)$.)

Is there a VA at $x = 2$? Yes, because $\lim_{x \to 0^+} f(x) = +\infty$.

EXAMPLE 10.12 (Intervals of Continuity). Determine the largest intervals of continuity for

$$f(x) = \begin{cases} x + 2 & \text{if } x \geq 3 \\ x^2 - x - 1 & \text{if } x < 3 \end{cases}.$$

Does f have an RD at $x = 3$? VA?

SOLUTION. The piecewise function consists of two polynomials. On $(3, \infty)$, $f(x) = x + 2$ is a continuous polynomial. Is f right-continuous at 3? Well, we know that $f(3) = 3 + 2 = 5$. And

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x + 2 = 5$$

So f is right continuous at 3. So one of the intervals of continuity is $[3, \infty)$.

On the interval $(-\infty, 3)$, $f(x) = x^2 - x - 1$ is polynomial and is continuous by Theorem 8.1. Is f left-continuous at 3?

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 - x - 1 \overset{\text{Poly}}{=} 5 = f(3).$$

So f is left-continuous at 3. Therefore, f is continuous on $(-\infty, 3]$. In fact, because the two one-sided limits at 3 are equal,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 5,$$

we have $\lim_{x \to 3} f(x) = 5 = f(3)$. So f is continuous at 3 (and everywhere else since it is a polynomial if $x \neq 3$. So $f(x)$ is actually continuous for all x, i.e., on $(-\infty, \infty)$.

Since f is continuous at $x = 3$ it does not have an RD [because $\lim_{x \to 3} f(x) = f(3) = 5$] and does not have a VA since the one-sided limits were not infinite.

Functions with Roots

So far we have not mentioned root functions or fractional power functions in our list of continuous functions. Our earlier fractional power limit property said

Assume that m and n are positive integers and that $\frac{n}{m}$ is reduced. Then

$$\lim_{x \to a} [f(x)]^{n/m} = \left[\lim_{x \to a} f(x) \right]^{n/m},$$

provided that $f(x) \geq 0$ for x near a if m is even.

So if when f is actually continuous at a, then this limit may be calculated as

$$\lim_{x \to a} [f(x)]^{n/m} = \left[\lim_{x \to a} f(x) \right]^{n/m} \overset{\text{Cont}}{=} [f(a)]^{n/m}.$$

This means that $[f(x)]^{n/m}$ is actually continuous at a, *provided that $f(x) \geq 0$ for x near a if m is even*.

One problem that is sometimes encountered is that f is continuous at a with $f(a) = 0$ and f is positive on one side of a and negative on the other and m is even. Under these circumstances, the analysis we just went through does not apply, because $[f(x)]^{n/m}$ will not be defined on one side of a. In these cases, we often find that $[f(x)]^{n/m}$ is either right- or left-continuous at a. The following theorem summarizes these ideas.

THEOREM 10.5 (Roots and Continuity). Assume that m and n are positive integers and that $\frac{n}{m}$ is in reduced form.

(a) If m is odd, then $[f(x)]^{n/m}$ is continuous at all points where f is continuous.

(b) If m is even, then $[f(x)]^{n/m}$ is continuous at all points a where f is continuous and $f(a) > 0$. (Note: Under these circumstances, f may be left- or right-continuous at any points a where f is continuous and $f(a) = 0$. These points should be considered separately.)

The following example illustrates the idea.

EXAMPLE 10.13 (Continuity and Roots). Determine the largest intervals of continuity for $f(x) = \sqrt{1 - x^2}$.

SOLUTION. You should recognize this function as the equation of the upper unit semi-circle. In any event, $f(x)$ is only defined where $1 - x^2 \geq 0$, i.e., where $1 \geq x^2$. So the domain of f is $-1 \leq x \leq 1$. Now we know that $1 - x^2$ is continuous since it is a polynomial. And by the root property for limits,

$$\lim_{x \to a} \sqrt{1 - x^2} \overset{\text{Root}}{=} \sqrt{\lim_{x \to a} 1 - x^2} \overset{\text{Poly}}{=} \sqrt{1 - a^2} = f(a)$$

whenever $1 - a^2 > 0$. So f is continuous *at least* on the open interval $(-1, 1)$. But what about the endpoints?

At $x = 1$, f is only defined on the left side of 1. Calculate the left-hand limit, making use of the one-sided limit property for fractional powers.

$$\lim_{x \to 1^-} \sqrt{1 - x^2} \overset{\text{Frac Pow}}{=} \sqrt{\lim_{x \to 1^-} 1 - x^2} \overset{\text{Poly}}{=} \sqrt{0} = 0.$$

Since $f(1) = 0$ also, then f is left-continuous at $x = 1$.

Similarly at $x = -1$, f is only defined on the right side of -1. This time

$$\lim_{x \to -1^+} \sqrt{1 - x^2} \overset{\text{Frac Pow}}{=} \sqrt{\lim_{x \to -1^+} 1 - x^2} \overset{\text{Poly}}{=} \sqrt{0} = 0.$$

Since $f(-1) = 0$, then f is left-continuous at $x = 1$. In total, then f is continuous on the closed interval $[-1, 1]$.

EXAMPLE 10.14 (Continuity and Roots). Determine the largest intervals of continuity for $f(x) = \sqrt[3]{x^4 - x + 1}$.

SOLUTION. Since $x^4 - x + 1$ is a polynomial, it is continuous everywhere. Since the root is odd, we can apply part (a) of Theorem 10.5 and conclude that $f(x) = \sqrt[3]{x^4 - x + 1}$ is continuous for all x.

More Continuous Functions

Many of the other basic functions that you have used in previous mathematics courses are also continuous.

THEOREM 10.6 (Trig Limits). The standard trig functions are continuous at all points in their domains. Specifically

1. $\lim_{x \to a} \sin x = \sin a$ and $\lim_{x \to a} \cos x = \cos a$ for all real numbers a.

2. $\lim_{x \to a} \sec x = \sec a$ and $\lim_{x \to a} \tan x = \tan a$ for all real numbers $a \neq, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$.

3. $\lim_{x \to a} \csc x = \csc a$ and $\lim_{x \to a} \cot x = \cot a$ for all real numbers $a \neq 0, \pm\pi, \pm2\pi, \pm3\pi, \dots$.

EXAMPLE 10.15. Here are a few examples of limits involving trig functions and other limit properties.

(a) $\lim_{x \to 1/2} \sin \frac{\pi x}{2} \stackrel{\text{Composite}}{=} \sin\left(\lim_{x \to 1/2} \frac{\pi x}{2} \right) \stackrel{\text{Poly}}{=} \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

(b) $\lim_{t \to 0}(t^2 + 4)\cos t \stackrel{\text{Product}}{=} \lim_{t \to 0}(t^2 + 4) \cdot \lim_{t \to 0}\cos t \stackrel{\text{Poly, Trig}}{=} 4 \cdot 1 = 4$.

(c) $\lim_{x \to \pi/4} \frac{\tan x}{x} \stackrel{\text{Quotient}}{=} \frac{\lim_{x \to \pi/4}\tan x}{\lim_{x \to \pi/4} x} \stackrel{\text{Trig, Poly}}{=} \frac{\tan \frac{\pi}{4}}{\frac{\pi}{4}} = \frac{1}{\frac{\pi}{4}} = \frac{4}{\pi}$.

(d) $\lim_{x \to \pi/2} \sin[\cos(x)] \stackrel{\text{Composite}}{=} \sin\left(\lim_{x \to \pi/2} \cos x \right) \stackrel{\text{Trig}}{=} \sin(\cos(\pi/2)) = \sin 0 = 0$.

(e) $\lim_{x \to 1} \cos(x^2 - 2x + 1) \stackrel{\text{Composite}}{=} \cos\left(\lim_{x \to 0} x^2 - 2x + 1 \right) \stackrel{\text{Trig}}{=} \cos(0) = 1$.

(f) $\lim_{x \to 0} \frac{1 - \cos x}{1 - \sqrt{\cos x}} = \lim_{x \to 0} \frac{1 - \cos x}{1 - \sqrt{\cos x}} \cdot \frac{1 + \sqrt{\cos x}}{1 + \sqrt{\cos x}} = \lim_{x \to 0} \frac{(1 - \cos x)(1 + \sqrt{\cos x})}{1 - \cos x} =$
$\lim_{x \to 0} 1 + \sqrt{\cos x} \stackrel{\text{Trig, Root}}{=} 2$.

If you look at your calculator, you will see keys for another handful of functions that we have not yet discussed: various logs[1] and exponentials. Along with the trig functions these are known as 'transcendental functions' because the transcend the ordinary algebraic operations of addition, subtraction, multiplication, and division, powers, and roots. There are a couple of reasons these transcendental functions appear on most calculators. First they are useful(!), and second they are 'nice.' In particular they are continuous. Specifically

THEOREM 10.7 (Logs and Exponentials). The standard log and exponential functions are continuous at all points in their domains. Specifically

1. $\lim_{x \to a} b^x = b^a$ for any real number $b > 0$ and for all real numbers a.

2. In particular, $\lim_{x \to a} e^x = e^a$ for all real numbers a.

3. $\lim_{x \to a} \ln x = \ln a$ for all real numbers $a > 0$. Similarly, $\lim_{x \to a} \log_b x = \log_b a$.

EXAMPLE 10.16. Again, these new limit properties may be combined with the previous limit properties to simplify the calculation of complicated-looking limits.

(a) $\lim_{x \to 3} \ln(x^2 + 1) \stackrel{\text{Composite}}{=} \ln\left(\lim_{x \to 3} x^2 + 1 \right) \stackrel{\text{Poly}}{=} \ln 10$.

(b) $\lim_{t \to 1} e^{t^2 - 1} \stackrel{\text{Composite}}{=} e^{\lim_{t \to 1}(t^2 - 1)} \stackrel{\text{Poly}}{=} e^0 = 1$.

(c) $\lim_{x \to 2} x^3 3^x \stackrel{\text{Product}}{=} \left(\lim_{x \to 2} x^3 \right) \cdot \left(\lim_{x \to 2} 3^x \right) \stackrel{\text{Poly, Exp}}{=} 2^3 \cdot 3^2 = 72$.

(d) $\lim_{x \to e} \ln x^5 = \lim_{x \to e} 5 \ln x \stackrel{\text{Log}}{=} 5 \ln e = 5$.

Take-home Message: Most of the familiar functions are continuous at all points in their domains. This includes: polynomials, rational functions, roots, trig, log, and exponential functions. Consequently, using Theorem 10.4, their composites are continuous as well.

[1] Note: When we use the term 'log' in this course, we will almost always mean the natural logarithm. OK, let's take a moment to review some basic log properties.

1. $\ln x^r = r \ln x$.

2. $\ln(xy) = \ln x + \ln y$.

3. $\ln \frac{x}{y} = \ln x - \ln y$.

4. $\ln e^x = x$ and $e^{\ln x} = x$ (inverse functions).

The Intermediate Value Theorem

Suppose that we want to solve an equation of the form $f(x) = k$. (Often we want to find the root(s) of a function, so we want to solve $f(x) = 0$.) Sometimes it is hard to find an exact solution, but nonetheless under certain circumstances we can tell that such a solution must exist (even if we don't know what it is). If f is continuous, the Intermediate Value Theorem below can be helpful in many situations.

THEOREM 10.8 (IVT: The Intermediate Value Theorem). Assume that f is continuous on the closed interval $[a, b]$ and that k is a number between $f(a)$ and $f(b)$. Then there is at least one number c in (a, b) so that $f(c) = k$.
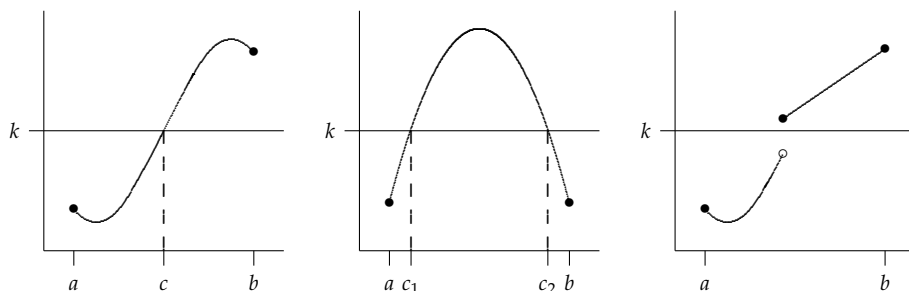

Figure 10.6: Left and middle: If f is continuous on the closed interval $[a, b]$ with k between $f(a)$ and $f(b)$, then there is at least one point c between a and b such that $f(b) = k$. Right: When f is not continuous, such a point c need not exist.

Though this theorem may seem 'obvious', the proof is surprisingly difficult and is covered in Math 331. Nonetheless, you can see how the hypothesis that f is continuous is critical. See the right-hand graph in Figure 10.6. When f is not continuous, the curve can 'jump over' the value k so that there is no point c in (a, b) with $f(c) = k$.

> **EXAMPLE 10.17.** Prove that $p(x) = 6x^4 + 4x^3 - 2x^2 - x - 3$ has a root in the interval $[-1, 1]$.
>
> **SOLUTION.** We want to solve $p(x) = 0$. Well, we probably are not going to be able to factor this polynomial. So let's see how we can apply the IVT, Theorem 10.8. Since p is a polynomial, it is continuous on the interval $[-2, 0]$. Moreover, at the endpoints $p(-1) = 6 - 4 - 2 + 1 - 3 = -2$ and $p(1) = 6 + 4 - 2 - 1 - 3 = 4$. Notice that 0 is between $p(-1) = -2$ and $p(1) = 4$. So by the IVT, there is some number c in $(-1, 1)$ so that $p(c) = 0$. We don't know the value of c, just that it exists. Neat!! (This is why theorems like the IVT are sometimes called **existence theorems**.)
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> **EXAMPLE 10.18.** Prove that $f(x) = x^3 - 4x + \cos(\pi x)$ has a root in the interval $[0, 1]$.
>
> **SOLUTION.** We want to solve $f(x) = 0$. Well, we can't factor this function. But we can apply the IVT, Theorem 10.8. First, $\cos(\pi x)$ is a composite of trig function and a polynomial, πx. Next, $x^3 - 4x$ is a polynomial and so is continuous. So f is the sum of continuous functions and so it is continuous on the interval $[0, 1]$. Moreover, at the endpoints $f(0) = 0 - 0 + 1 =$ and $f(1) = 1 - 4 - 1 = -4$. Notice that 0 is between $f(0) = 1$ and $f(1) = -4$. So by the IVT, there is some number c in $(0, 1)$ so that $f(c) = 0$. Again, we don't know the value of c, just that it exists!!

YOU TRY IT 10.8 (Hand in for extra credit). . Show that $p(x) = x^4 - x^3 + x^2 + x - 1$ has at least two roots in $[-1, 1]$. Big hint: Split $[-1, 1]$ in half into two smaller closed intervals. Show that p has a root in each of the two smaller intervals.

> **EXAMPLE 10.19.** Your parents invest $20,000 in a savings account for you when you are 8-years-old. They want it to be worth $50,000 ten years later when when you start HWS. If the account has an annual interest rate r, with monthly compounding, Then the amount in the account after 10 years (120 months) is
>
> $$A(r) = 20,000 \left(1 + \frac{r}{12}\right)^{120}.$$