2.4 Limits

What we need is some way to make this notion of 'approaching' that we used in Example 2.1 more precise. In a few days we will give a technical definition. But for the moment, we will use the following idea.

DEFINITION 2.1 (Informal Definition of Limit). Suppose that the function f is defined for all x near a, except perhaps at a. We write

$$\lim_{x \to a} f(x) = L$$

and say that **the limit of** $f(x)$ **as** x **approaches** a **is** L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to (but not equal to) a.

Note: In this definition, x is assumed to be approaching a from both the left and right sides of a.

EXAMPLE 2.4. (Finding limits from a graph) Use the graph of f in Figure 2.9 to determine the following values, if possible.

(a) $f(1)$ and $\lim_{x \to 1} f(x)$ (b) $f(3)$ and $\lim_{x \to 3} f(x)$

(c) $f(4)$ and $\lim_{x \to 4} f(x)$ (d) $f(5)$ and $\lim_{x \to 5} f(x)$

SOLUTION. See the graphs below in Figure 2.10.

(a) Notice that we need to focus only on the section of the graph near $x = 1$. We see that $f(1) = 0.8$. As x approaches $x = 1$ from either side (see arrows), the values of $f(x)$ approach 0.8. So $\lim_{x \to 1} f(x) = 0.8$
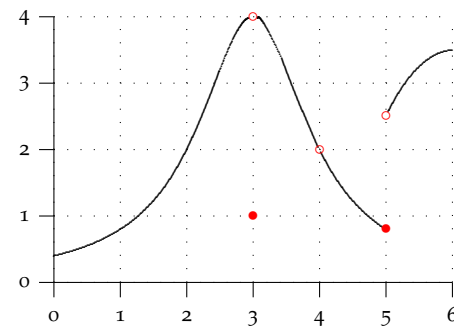
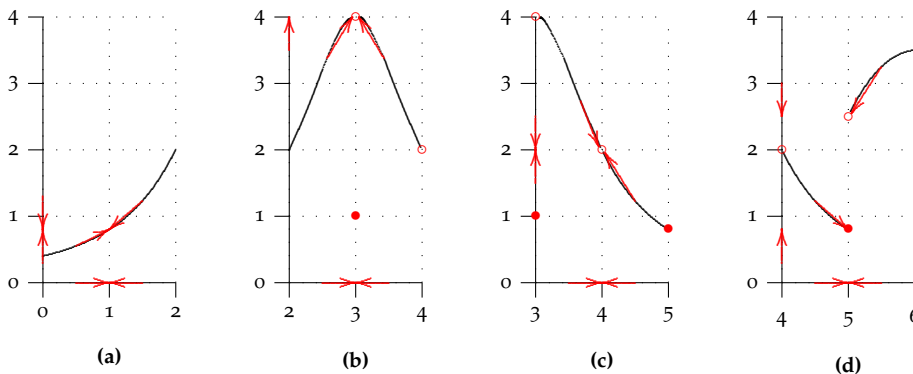Figure 2.9: Determine the function values and limits, if they exist.

Figure 2.10: Determine the indicated function values and limits, if they exist.

(a)   (b)   (c)   (d)

(b) We focus only on the section of the graph near $x = 3$. We see that $f(3) = 1$ (solid dot). But as x approaches $x = 3$ from either side, the values of $f(x)$ approach 4 (hollow dot). So $\lim_{x \to 3} f(x) = 4$. The actual value of the function right at $x = 3$ does not matter when computing the limit. For the limit, we want to know what the function is *approaching*, not what its value *is*.

(c) We now focus on the section of the graph near $x = 4$. Notice that $f(4)$ is not even defined (hollow dot). But as x approaches $x = 4$ from either side, the values of $f(x)$ approach 2 (hollow dot). So $\lim_{x \to 4} f(x) = 2$ *even though* $f(4)$ *does not exist*. Again, for the limit, we want to know what the function is *approaching*, not what its value *is*.

(d) We focus on the graph near $x = 5$. We see that $f(5) = 0.8$ (solid dot). And as x approaches $x = 3$ from the left side, the values of $f(x)$ also approach 0.8

(solid dot). But as x approaches $x = 5$ from the right side, the values of $f(x)$ approach 2.5 (hollow dot). In other words, the function is approaching *different* values from either side and no limit exists: $\lim_{x \to x} f(x)$ DNE. The actual value of the function right at $x = 3$ does not matter when computing the limit.

Take-home Message.   The value of $\lim_{x \to a} f(x)$, if it exists, depends on the values of f on both sides of a—but not on the value of f right at a itself. For the limit to exist at a, the values of f must approach the same number on both sides of a.

YOU TRY IT 2.2. Here are a few quick problems to check your understanding of limits.

(*a*) Use Figure 2.9 to determine $\lim_{x \to 2} f(x)$.

(*b*) Suppose that we changed the positions of the hollow and solid dots at $x = 5$. Would that change the limit?

(*c*) Suppose that $f(4) = 4$. Would that change the limit?

(*d*) What can you say about $\lim_{x \to 6} f(x)$?

EXAMPLE 2.5. (Finding limits with algebra) Return to Example 2.1 where $f(x) = x^2 - 2x$. We wanted to find the tangent slope at $x = 3$. As $x \to 3$, $m_{sec} \to m_{tan}$ or using limits we can write $\lim_{x \to 3} m_{sec} = m_{tan}$. We see that the notion of limit is exactly what we need. We wanted to determine what happened to the difference quotient function

$$\frac{f(x) - f(3)}{x - 3}$$

as x got close to 3. In terms of Definition 2.1, we wanted to compute

$$\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}.$$

In fact, we were able to evaluate this limit because

$$\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 1)}{x - 3} = \lim_{x \to 3} (x + 1) = 4.$$

At the last step, we can say that as x gets close to 3, $x + 1$ gets close to 4. Notice that we never simply 'plugged' 3 into the function $\frac{f(x) - f(3)}{x - 3}$ since it is undefined at $x = 3$. That's why the limit definition says the limit is evaluated 'by taking x sufficiently close to (but not equal to) a.'

When we graph

$$\frac{f(x) - f(3)}{x - 3} = \frac{x^2 - 2x - 3}{x - 3} = x + 1 \qquad (x \neq 3)$$

we can see geometrically why the limit is 4. The graph is a line with a hole at $x = 3$. The line approaches 4 as the x approaches (but is not equal to) 3. Neat!

EXAMPLE 2.6. (Finding limits with a table) Determine $\lim_{x \to 0} \dfrac{\sin x}{x}$, if it exists. Notice that *this function is not even defined at 0*. Does that matter?

SOLUTION. Notice that as x gets close to 0 both the numerator and denominator approach 0. However, there is no algebra that we can carry out to simplify the function. We could make a graph of this function using a calculator and proceed as in Example 2.4. But there is another approach that can allow us to *estimate* what the limit might be. We can take values of $f(x) = \frac{\sin x}{x}$ near $x = 0$ and on both side of 0. By arranging them in a table, we can see what happens to $f(x)$. Take a moment to fill in the missing values. Do you think the limit exists? If so, can you estimate its value?

You should have found that the function values in both columns were the the same. From these values it appears that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$. This is pretty interesting since the function is not even defined at $x = 0$.
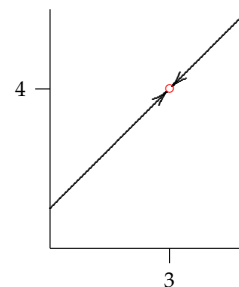

Figure 2.11: The function

$$\frac{f(x) - f(3)}{x - 3} = x + 1 \quad (x \neq 3).$$

$g(x)$ is not defined at $x = 3$, nonetheless $\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = 4$. The values approach 4 as x approaches (but is not equal to) 3.

$x < 0$	$\frac{\sin x}{x}$	$x > 0$	$\frac{\sin x}{x}$
-0.5	0.95885108	0.5	
-0.1	0.99833417	0.1	
-0.01		0.01	0.99998333
-0.001		0.001	

EXAMPLE 2.7 (From Physics). A ball is thrown upwards from a height of 20 m. Its position (height) above the ground after t seconds is given by[2]

[2] A more accurate equation would be $f(t) = 20 + 10t - 4.9t^2$.

$$f(t) = 20 + 10t - 5t^2 \qquad (0 \le t \le 3)$$

(a) Here's a very general problem: Find the average velocity of the ball on the time interval $[a, t]$.

$$\text{Ave vel on } [a, t] = m_{\text{sec}} = \frac{f(t) - f(a)}{t - a} = \frac{20 + 10t - 5t^2 - (20 + 10a - 5a^2}{t - a}$$
$$= \frac{10t - 10a - 5t^2 + 5a^2}{t - a}$$
$$= \frac{10(t - a) - 5(t^2 - a^2)}{t - a}$$
$$= \frac{10(t - a) - 5(t + a)(t - a)}{t - a}$$
$$= 10 - 5(t + a) \, \text{m/s}. \qquad (2.11)$$

(b) Now we begin to make the problem more specific: Find the average velocity on the interval $[1, t]$. This time we don't need to go through the calculation of the difference quotient as above. We may simply use the formula we developed in part (a). We let $a = 1$ and substitute into (2.11).

$$\text{Ave vel on } [1, t] = m_{\text{sec}} = 10 - 5(t + 1) = 5 - 5t \, \text{m/s}.$$

(c) More specifically still: Find the average velocity on the interval $[1, 2]$. This time we let $a = 1$ and $t = 2$ and substitute into (2.11).

$$\text{Ave vel on } [1, 2] = m_{\text{sec}} = 10 - 5(2 + 1) = -5 \, \text{m/s}.$$

How would do interpret the fact that the average velocity is negative?

(d) Find the average velocity on the interval $[0, 1]$. This time we let $a = 1$ and $t = 0$ and substitute into (2.11).

$$\text{Ave vel on } [0, 1] = m_{\text{sec}} = 10 - 5(0 + 1) = 5 \, \text{m/s}.$$

(e) If we wanted to find an approximation to the instantaneous velocity right at $a = 1$, we could take a very small time interval, say $[1, 1.001]$ and find the average velocity.

$$\text{Ave vel on } [1, 1.001] = m_{\text{sec}} = 10 - 5(1.001 + 1) = -.005 \, \text{m/s}.$$

(f) To obtain the instantaneous velocity right at $a = 1$, we take the limit of the average velocity on the interval $[1, t]$ as $t \to 1$.

$$\text{Inst vel at } 1 = m_{\text{tan}} = \lim_{t \to 1} \text{Ave vel on } [1, t] = \lim_{t \to 1}[10 - 5(t + 1)] = 10 - 10 = 0 \, \text{m/s}.$$

(g) What does it mean to say that the instantaneous velocity at time $t = 1$ is 0 m/s? One interpretation is that the object is at an 'instantaneous standstill.' Think about the flight of the ball: It is tossed upwards and then falls to the ground. At some point...the top of its flight...it must stop going up and then begin to fall. At the very top of its flight the velocity is 0. Geometrically, this means that the tangent slope of the curve right at $t = 1$ is 0. In other words, the tangent line is horizontal as we would expect at the top of this parabola. See Figure 2.12.
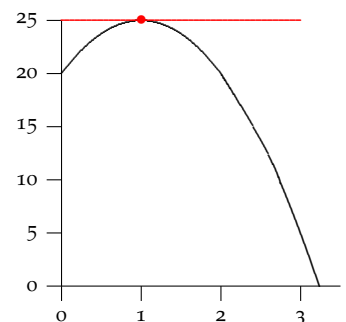

Figure 2.12: The tangent line to $f(t) = 20 + 10t - 5t^2$ has is horizontal. This

Take-home message.   In a few days, we will discus a precise method for finding limits. Graphs and tables are helpful, but are occasionally misleading. But they are the best tools available to us for the moment. The most interesting limits occur when the function value is either not defined at the point of interest or is different from the limit value.

YOU TRY IT 2.3. Review the limit concept in Definition 2.1. Then use the graph of f below to determine the function values and limits below, if they exist. Remember what • and ○ indicate. Are the first answers correct?
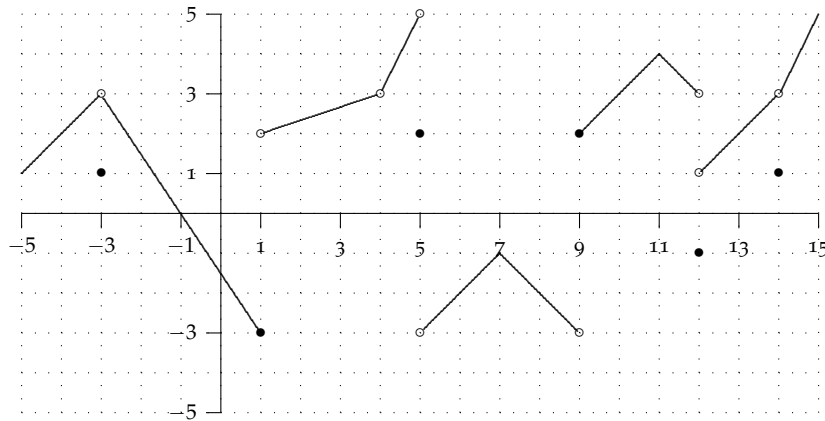
(a) What is $f(-3)$ Answer: 1   What is $\lim_{x \to -3} f(x)$ Answer: 3

(b) What is $f(0)$ What is $\lim_{x \to 0} f(x)$

(c) What is $f(1)$ What is $\lim_{x \to 1} f(x)$

(d) What is $f(4)$ What is $\lim_{x \to 4} f(x)$

(e) What is $f(5)$ What is $\lim_{x \to 5} f(x)$

(f) What is $f(7)$ What is $\lim_{x \to 7} f(x)$

(g) What is $f(9)$ What is $\lim_{x \to 9} f(x)$

(h) What is $f(11)$ What is $\lim_{x \to 11} f(x)$

(i) What is $f(12)$ What is $\lim_{x \to 12} f(x)$

(j) What is $f(13)$ What is $\lim_{x \to 13} f(x)$

(k) What is $f(14)$ What is $\lim_{x \to 14} f(x)$

2.5   One-Sided Limits

We have made a point of saying that $\lim_{x \to a} f(x) = L$ only if $f(x)$ approaches L as x approaches a from *both* sides of a. For this reason, $\lim_{x \to a} f(x) = L$ is sometimes referred to as the **two-sided limit** of f at a. In some instances it makes sense to talk about limits from one side or the other of a.
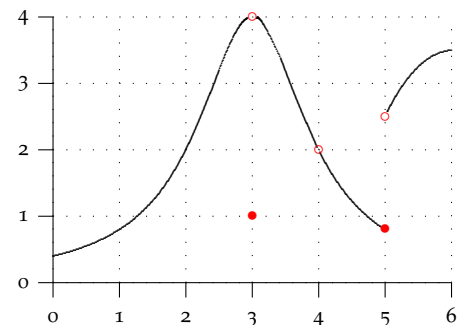

Figure 2.13: What can we say about the limit of f as x approaches 6?

Look again at Figure 2.13 which we examined earlier. In YOU TRY IT 2.2 we asked what you could say about $\lim_{x \to 6} f(x)$. Since the graph only exists on one side of 6 the (two-sided) limit does not exist. Nonetheless, we clearly see that as x approaches 6 from the left (or 'from below'), $f(x)$ approaches 3.5. So we can say *something* intelligent about f near 6, even though the two-sided limit does not exist. We need some new language to describe this.

DEFINITION 2.2 (One-Sided Limits). Assume that f is defined for all x near a with $x > a$. We write

$$\lim_{x \to a^+} f(x) = L$$

and say that **the right-hand limit** of $f(x)$ as x approaches a is L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to but greater than a.

Assume that f is defined for all x near a with $x < a$. We write

$$\lim_{x \to a^-} f(x) = L$$

and say that **the left-hand limit** of $f(x)$ as x approaches a is L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to but smaller than a.

$\lim_{x \to a^+} f(x)$ is also called the **limit from above** and $\lim_{x \to a^-} f(x)$ is also called the **limit from below**.

EXAMPLE 2.8. Using Figure 2.13 determine the following limits, if they exist.

(a) $\lim_{x \to 3^-} f(x)$ (b) $\lim_{x \to 3^+} f(x)$ (c) $\lim_{x \to 5^-} f(x)$ (d) $\lim_{x \to 5^+} f(x)$

SOLUTION. Careful: These are all one-sided limits.

(a) As x approaches 3 from the left (or below), notice that the values of f approach 4, even though $f(4) = 1$. So $\lim_{x \to 3^-} f(x) = 4$.

(b) Similarly, as x approaches 3 from the right (or above), the values of f approach 4, even though $f(4) = 1$. So $\lim_{x \to 3^+} f(x) = 4$. In this case the left- and right-hand limits at 3 were equal.

(c) This time, as x approaches 5 from the left (or below), the values of f approach 0.8 so $\lim_{x \to 5^-} f(x) = 0.8$.

(d) However, as x approaches 5 from the right (or above), the values of f approach 2.5 so $\lim_{x \to 5^+} f(x) = 2.5$. The two one-sided limits at $x = 5$ are different. We say that the function $f(x)$ has a **jump** at $x = 5$.

The two points exhibited different behavior. At $x = 3$, both one-sided limits were equal to 4. That's why we were able to say earlier in Example 2.4 that the two-sided limit existed and that $\lim_{x \to 3} f(x) = 4$. On the other hand, because the two one-sided limits are different at 5, this means that f approaches two different numbers as x gets close to 5. Hence the two sided limit there does not exist, that is, $\lim_{x \to 5} f(x)$ DNE. This observation leads to the following theorem.

THEOREM 2.1 (One- and Two-Sided Limits). Assume that the function f is defined for all x near a, except perhaps at a. Then $\lim_{x \to a} f(x) = L$ if and only if both $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

Interpretation. If the two sided limit exists, then so do both one-sided limits and they are both equal to the two-sided limit. If the two one-sided limits are *not equal* or if one or the other does not exist, then the two-sided limit does not exist.

YOU TRY IT 2.4. Using Figure 2.13 determine the following limits, if they exist.

(a) $\lim\limits_{x \to 2^-} f(x)$ (b) $\lim\limits_{x \to 2^+} f(x)$ (c) $\lim\limits_{x \to 2} f(x)$

(d) $\lim\limits_{x \to 4^-} f(x)$ (e) $\lim\limits_{x \to 4^+} f(x)$ (f) $\lim\limits_{x \to 4} f(x)$

(g) Which limit can you evaluate at $x = 0$? What is its value?

(h) Which limit can you evaluate at $x = 6$? What is its value?

Answers to YOU TRY IT 2.4 : (a)–(f) 2; (g) $\lim_{x \to 0^+} f(x) = 0.5$; (g) $\lim_{x \to 6^-} f(x) = 3.5$.

EXAMPLE 2.9 (Wild Behaviour!). Try to determine $\lim\limits_{x \to 0} \sin\frac{\pi}{x}$ by filling in the table of values below.

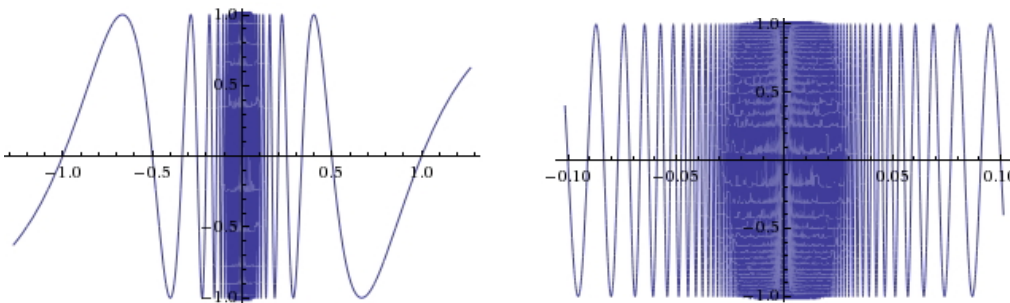**SOLUTION.** Notice that $\sin\frac{\pi}{x}$ is not defined at 0.

$x < 0$	$\sin\frac{\pi}{x}$	$x > 0$	$\sin\frac{\pi}{x}$
-0.5	0	0.5	0
-0.1	0	0.1	0
-0.01	0	0.01	0
-0.0011		0.0011	-0.28173
-0.00021		0.00012	
-0.000013		0.000023	
-0.0000024	0.781831	0.0000014	

From the first three rows of the table, we might believe that $\lim\limits_{x \to 0} \sin\frac{\pi}{x} = 0$. But if we put in rather more arbitrary numbers we see that we get some very different values. Take the time to fill in the rest of the table. It does not appear that there is any limit.

Why did we get a bunch of 0's and then different values? Well, if we let $x = \frac{1}{n}$ where n is any non-zero integer, then

$$\sin\frac{\pi}{x} = \sin\frac{\pi}{\frac{1}{n}} = \sin n\pi = 0.$$

In the first few table entries we put in $\pm\frac{1}{2}$, $\pm\frac{1}{10}$, and $\pm\frac{1}{100}$, which will all produce 0's. But letting x be a truly arbitrary small number yields non-zero values between -1 and 1. Since $\sin\frac{\pi}{x}$ does not approach a single number either from the left or the right, neither one-sided limit exists, so the two-sided limit does not exist either.   If


Figure 2.14: Left: The graph of $f(x) = \sin\frac{\pi}{x}$ on $[-1, 1]$. Right: A close-up of the graph of $f(x) = \sin\frac{\pi}{x}$ on $[-0.1, 0.1]$. What is $\lim\limits_{x \to 0} \sin\frac{\pi}{x}$?

we actually tried to plot the graph of $\sin\frac{\pi}{x}$, we see the wild oscillating behavior as x approaches 0 from either side as in Figure 2.14. Try this same plot on your calculator. What happens? Does the graph look the same?

Conclusion.   Until we encountered Example 2.9 using graphs and tables worked reasonably well to compute the limits we examined. But Example 2.9 points out

the problems that can arise with these methods. We may happen to choose values for our table of values that are misleading or we may find that a calculator is unable to provide an accurate graph of the function near the point in question. For these reasons, whenever possible we will rely on algebraic techniques, as we did in Example 2.7. We will explore these methods in the next few sections.

YOU TRY IT 2.5. Use the graph below to answer the questions.
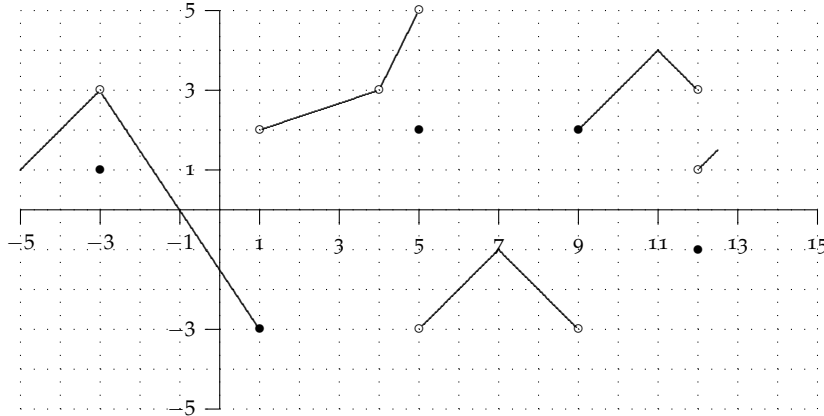
(*a*) Fill in the first 4 empty columns using the graph.

a	$f(a)$	$\lim\limits_{x \to a^a} f(x)$	$\lim\limits_{x \to a^+} f(x)$	$\lim\limits_{x \to a} f(x)$	Continuous
-4					
-3					
-1					
0					
1					
3					
4					
5					
9					
11					
12					

(*b*) A function f is **continuous** at a if $\lim\limits_{x \to a} f(x) = f(a)$, i.e., if the function's value and its limit both exist and are the same number. Use the table above (which columns?) to determine whether f is continuous at each of the indicated points (Yes/No).

(*c*) List five points other than those in the table between -4 and 12 where f is also continuous.

(*d*) Complete the last two columns of the table below. Then complete the graph above on the interval 12.5 to 15 using the information in the table.

a	$f(a)$	$\lim\limits_{x \to a^a} f(x)$	$\lim\limits_{x \to a^+} f(x)$	$\lim\limits_{x \to a} f(x)$	Continuous
13	2	2	2		
14	1	3	2		