The Precise Limit Definition

The Limit Definition

We are ready to make precise what we mean the term ‘limit.’ So far we have worked with an intuitive understanding of the term:

DEFINITION 5.1 (Informal Definition of Limit)**.** We write $\lim_{x \to a} g(x) = L$ and say that **the limit of** $g(x)$ **as** x **approaches** a if we can make $g(x)$ arbitrarily close to L by taking x sufficiently close to (but not equal to) a.

One of the points that distinguishes mathematics from other disciplines is the preciseness of its definitions. So compare the definition above to the

DEFINITION 5.2 (Formal Definition of Limit)**.** Let f be a function defined on some open interval containing a, except perhaps at a itself. We say that $\lim_{x \to a} f(x) = L$ if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ so that

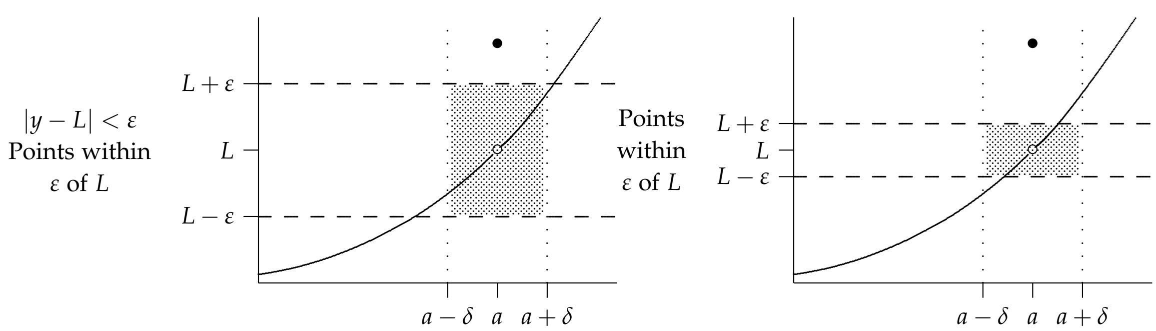
$$\text{if} \quad 0 < |x - a| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Each inequality makes precise some aspect of the informal definition of limit.

- $|f(x) - L| < \varepsilon$ means “we can make $f(x)$ arbitrarily close to L” (within ε of L)
- by taking $|x - a| < \delta$, i.e.,“taking x sufficiently close to a" (within δ of a)
- The inequality $0 < |x - a|$ simply means “x is not equal to a.”

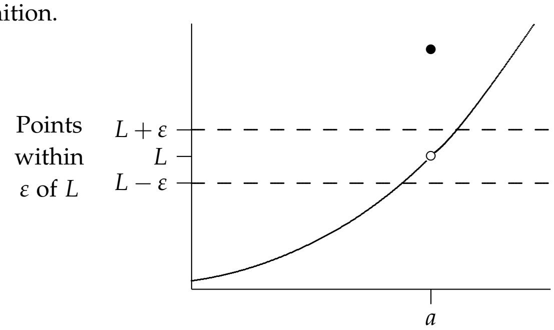
Examples of Using the Limit Definition

EXAMPLE 5.1. On the left: For the given ε, the selected δ keeps $f(x)$ within within the horizontal band, that is, within ε of L over the interval from $a - \delta$ to $a + \delta$ (except perhaps at a). On the right: With a smaller ε, the old δ may fail. Is it possible to find a new δ that works?



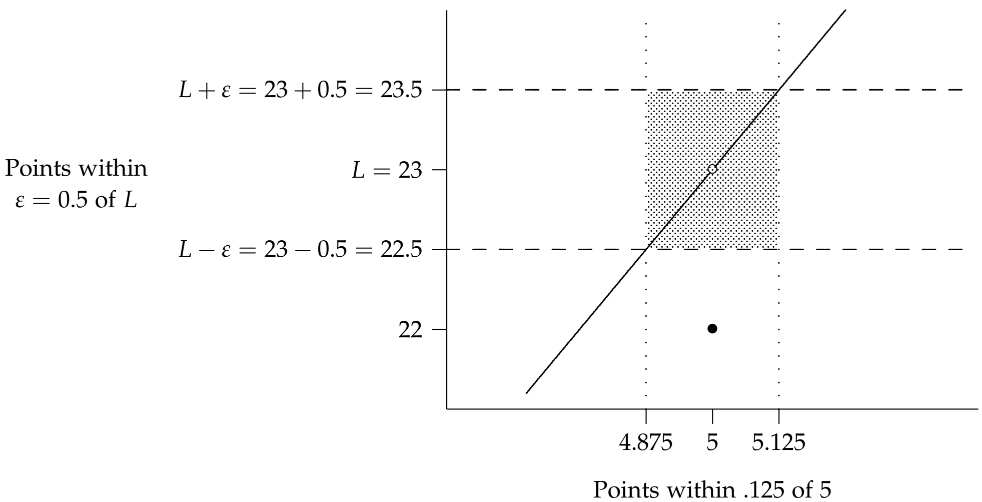
$|x - a| < \delta$: Points within δ of a Points within δ of a

EXAMPLE 5.2. The formal definition says that for each and every ε, we need to be able to find a δ. With the same function as before, for a smaller choice of ε, the earlier value of δ might not work. However: As long as we can find a new δ for each new ε, the limit will exist. For the smaller choice of ε, draw a δ interval about a which satisfies the limit definition.

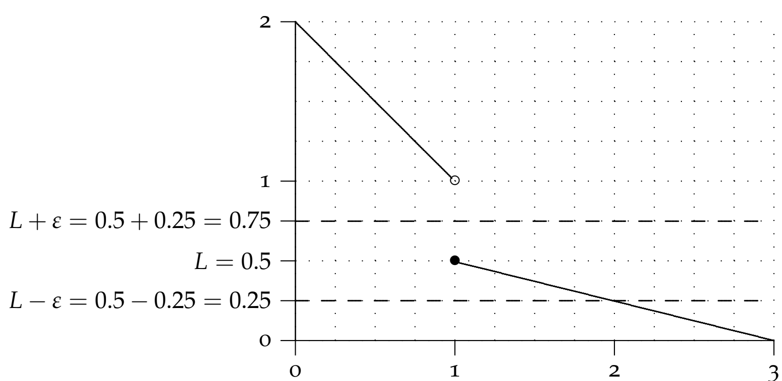


To repeat: As long as we can find a new δ for each new ε, the limit will exist.

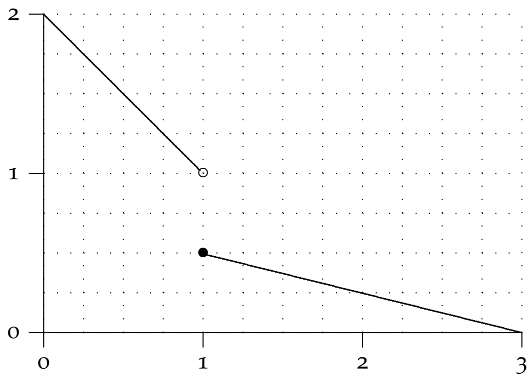
EXAMPLE 5.3. For the function $f(x) = \begin{cases} 4x+3, & \text{if } x \neq 5 \\ 22, & \text{if } x = 5 \end{cases}$. **Use absolute values to describe the shaded region.**



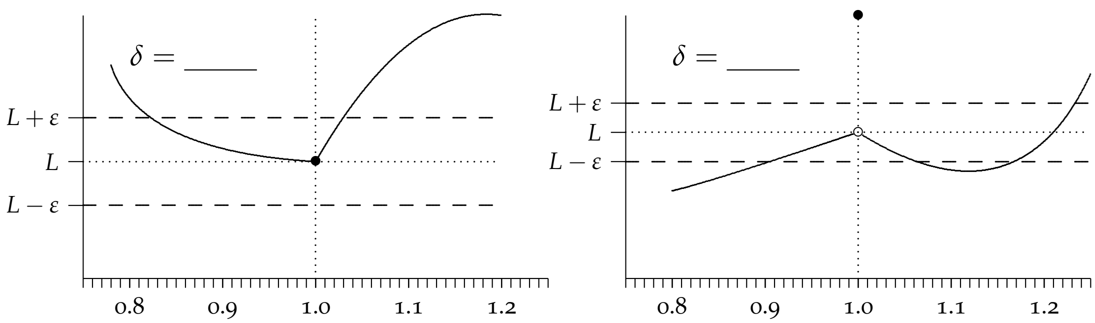
EXAMPLE 5.4. Let $f(x) = \begin{cases} 2-x, & \text{if } x \leq 1 \\ 1-\frac{1}{2}x, & \text{if } x > 1 \end{cases}$. Intuitively we can see that $\lim_{x\to 1} f(x)$ DNE. For example we show that $\lim_{x\to 1} f(x) \neq 0.5$ by using $\varepsilon = 0.25$. Now there is NO $\delta > 0$ that satisfies the limit definition.



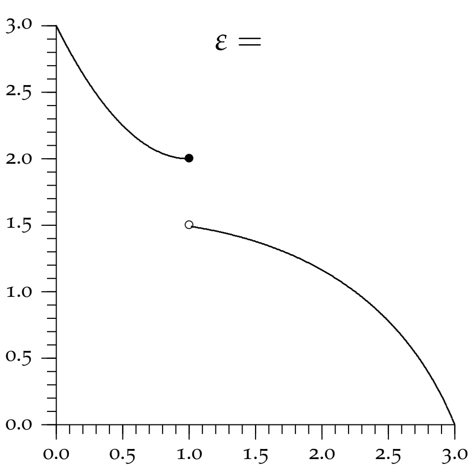
YOU TRY IT 5.1. ☞ Here's another copy of the graph of the function in Example 5.4. Find a particular value of ε to show that $\lim_{x\to 1} f(x) \neq 1$.



YOU TRY IT 5.2. In each figure below, for the given choice of ε, find and draw a δ interval (a vertical strip) about $a = 1$ which satisfies the limit definition. Note: The same δ must work on both the left and right at sides of $a = 1$. What is δ in each case? Note the scale.



YOU TRY IT 5.3. For the function $f(x)$ on the left below, show that that 2 is **not** $\lim_{x\to 1} f(x)$. To do this, **find and draw** a horizontal ε interval about $y = 2$ for which there is **no** value of δ that will satisfy the limit definition. What is the value of your choice of ε?



Is $f(x)$ continuous at $x = 1$? (Explain for each.)

Does $f(x)$ have a removable discontinuity at $x = 1$?

Does $\lim_{x\to 1^+} f(x)$ exist?

Does $\lim_{x\to 1^-} f(x)$ exist?

Is $f(x)$ continuous at $x = 2$?

Exercises: The Limit Applet

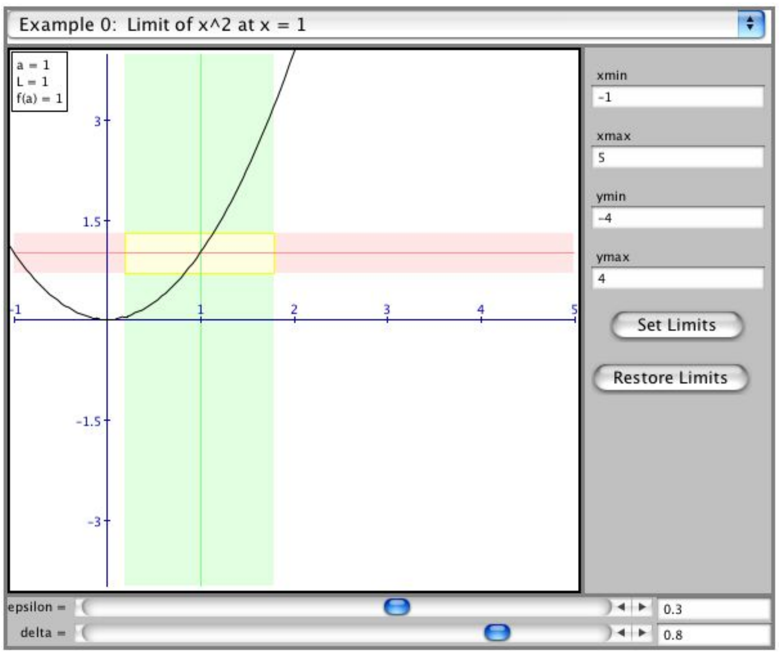
The formal epsilon/delta definition of the limit is:

$\lim_{x\to a} f(x) = L$ if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ so that

$$\text{if} \quad 0 < |x - a| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

1. With a partner, start up the limit applet at `http://math.hws.edu/~mitchell/Math130F12/Web/LimitsNew.html` or use the link at the course website.

2. On screen (and in the picture below), the graph of a function, $f(x)$, is shown (in black) and numbers a and L have been picked for you.
   (a) You can change the values of epsilon and delta by dragging the sliders at the bottom of the applet or by typing new numbers into the input boxes. Try it now.
   (b) On the graph, a green line represents the vertical line $x = a$, and a red line represents the horizontal line $y = L$.
   (c) The pink area, together with the yellow area, consists of the points (x, y) for which $|y - L| < \varepsilon$.
   (d) The light green area, together with the yellow area, consists of the points (x, y) for which $|x - a| < \delta$. You can vary the widths of these regions by changing the values of epsilon and delta. Try it.



3. What does it mean in this applet for the limit of $f(x)$ as x approaches a to be equal to L? It means that you can set the epsilon slider to any value you like, except zero. Then, no matter what positive epsilon you've chosen, it must be possible to adjust the delta slider so that the graph passes through the yellow rectangle without straying into the green region. (A point $(x, f(x))$ in the green region represents a value of x such that $|x - a| < \delta$ but $|f(x) - L| \geq \varepsilon$. This is bad.) Note that if epsilon is made smaller, then delta might also have to be made smaller. If the limit is L, you will always be able to make delta small enough to work. If you encounter an epsilon for which no value of delta will work, then the limit is not equal to L. (The limit might still exist, if it's equal to some value other than L.)
   (a) To reset the example to its original values by using the "Restore Limit" button or reselecting "Example 0: The Limit of x^2 at $a = 1$" using the menu bar at the top of the applet. The upper left corner of the applet shows the value of a, the suspected limit L and the function value of f at a, if $f(a)$ is defined. What are these three values for this example?

(*b*) In this example at first, the delta value is too big for the epsilon that is selected. The graph hits the green area. However, a smaller value of delta will work. Change delta to 0.1, for example. To get a better view, you can zoom in on a point by clicking on it. Click on the point where the red and green lines intersect two or three times. This will enlarge the graph nicely.

(*c*) Now, try a smaller value of epsilon. Set epsilon to 0.1. The value delta = 0.1 is now too big. But if you reduce it, say to delta = 0.04, it works again. Fill in the table

$a =$	$L =$		$f(a) =$	
Epsilon ε	0.3	0.01	0.05	0.001
Delta δ				
Conclusion: For each ε does there appear to be a corresponding δ? Does the limit exist?				

(*d*) Reduce epsilon again, say to 0.05. Find an appropriate delta.

(*e*) If this process can be repeated infinitely often—no matter how small epsilon is, you can find a delta—then the limit as x approaches a is L. **However**, if you ever get stuck—if you find a positive epsilon for which no positive delta works—then the limit as x approaches a is not L. On a computer, of course, you can't really repeat this process infinitely, since the computer can't deal with arbitrarily small numbers. But it can help you understand what is going on.

4. **Example 2: Gap**. Fill in the values of a, L, and $f(a)$, then try to find delta for each of the given epsilon values. Decide whether the limit exists and justify your answer.

$a =$	$L =$		$f(a) =$	
Epsilon ε	0.1	0.07	0.04	0.02
Delta δ				
Conclusion: For each ε does there appear to be a corresponding δ? Does the limit exist?				

☞ Before doing the homework, make sure that you understand the startup example! *Remember: For the limit to exist, for each epsilon, you need to find a delta so that the graph stays in the yellow and pink regions and does not enter the green region.* There are more examples, including the ones from class today at `http://math.hws.edu/~mitchell/Math130F12/Web/LimitsNew.html`

Limit Proofs: A Two-Step Process

This section demonstrates how mathematicians use the formal definition of limit to give careful proofs of limit calculations. We will look at some very simple examples that illustrate this idea.[1] We will use a two-stage process to prove that $\lim_{x\to a} f(x) = L$. Remember that

[1] To learn about more complex limit calculations, take Math 331.

$\lim_{x\to a} f(x) = L$ if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ so that

$$\text{if} \quad 0 < |x-a| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

So we need to

1. SCRAP WORK: Find δ. We do this by letting ε be an arbitrary positive number and then we use the inequality $|f(x) - L| < \varepsilon$ to work 'backwards' to a statement of the form $|x - a| < \delta$, where δ depends only on ε.
2. ARGUMENT: Write the proof. For any $\varepsilon > 0$, assume $0 < |x-a| < \delta$ and use the work in Step 1 to prove that $|f(x) - L| < \varepsilon$. (Here we work 'forward.')

The first step is essentially 'scrap work' for the proof. The second step is the actual 'proof' that our choice of δ works. This is clearer in an example.

EXAMPLE 5.5. (The Limit of a Linear function) Prove that $\lim_{x\to 2}(3x+5) = 11$ using the formal definition of limit.

SOLUTION. SCRAP WORK: Find δ. In this case $a = 2$ and $L = 11$. Assume that $\varepsilon > 0$ is given (but arbitrary). Work backwards:

$$|f(x) - L| < \varepsilon \overset{\text{Translate}}{\iff} |(3x+5) - 11| < \varepsilon \overset{\text{Simplify}}{\iff} |3x - 6| < \varepsilon$$
$$\overset{\text{Factor}}{\iff} 3|x-2| < \varepsilon \overset{\text{Solve}}{\iff} |x-2| < \frac{\varepsilon}{3}.$$

Notice at this last step, we have an inequality of the form $|x - a| < \delta$. We identify δ as $\frac{\varepsilon}{3}$. Notice how δ depends on ε. In particular, as ε gets smaller, so does δ. We saw this geometrically with applet and the graphs in the first part of this handout. Now we are ready to write the actual proof.

ARGUMENT: Let $\varepsilon > 0$ be given. Assume[2] that $0 < |x-2| < \frac{\varepsilon}{3}$. Then

[2] Remember, we are choosing $\delta = \frac{\varepsilon}{3}$. We are trying to show that

$$\text{if } 0 < |x-a| < \delta, \text{ then } |f(x) - L| < \varepsilon$$

or in this particular case

$$\text{if } 0 < |x-2| < \frac{\varepsilon}{3}, \text{ then } |(3x+5) - 11| < \varepsilon.$$

$$|(3x+5) - 11| \overset{\text{Simplify}}{=} |3x-6| \overset{\text{factor}}{=} 3|x-2| \overset{|x-2|<\frac{\varepsilon}{3}}{<} 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

(At this point, the proof is complete: We have shown that for any $\varepsilon > 0$, if $0 < |x - 2| < \delta = \frac{\varepsilon}{3}$, then $|(3x+5) - 11| < \varepsilon$. Having done the necessary scrap work, the entire proof consists of three short sentences.)

YOU TRY IT 5.4. Once we have found δ in terms of ε, no matter what ε we choose, an appropriate δ can be found. In Example 5.5, if $\varepsilon = 0.01$, what would δ be? Or if $\varepsilon = 0.0001$, how would you choose δ?

Also if $\delta = \frac{\varepsilon}{3}$ 'works,' then any smaller positive value of δ will also work. Show that $\delta = \frac{\varepsilon}{4}$ 'works' in the proof above.

Answers to YOU TRY IT 5.4 : $\delta = \frac{\varepsilon}{3} = \frac{0.01}{3}$, and $\delta = \frac{0.0001}{3}$.
This time $|3x - 6| = 3|x-2| < 3 \cdot \frac{\varepsilon}{4} < \varepsilon$.

EXAMPLE 5.6. (Another Linear function) Prove that $\lim_{x\to 1}(2-5x) = -3$ using the formal definition of limit.

SOLUTION. SCRAP WORK: Find δ. In this case $a = 1$ and $L = -3$. Assume that $\varepsilon > 0$ is given (but arbitrary). Work backwards:

$$|f(x) - L| < \varepsilon \overset{\text{Translate}}{\iff} |(2-5x) - (-3)| < \varepsilon \overset{\text{Simplify}}{\iff} |-5x+5| < \varepsilon$$
$$\overset{\text{Factor}}{\iff} |-5||x-1| < \varepsilon$$
$$\overset{\text{Solve}}{\iff} |x-1| < \frac{\varepsilon}{|-5|} = \frac{\varepsilon}{5}.$$

We have our inequality of the form $|x-a| < \delta$ with $\delta = \frac{\varepsilon}{5}$. Now write the actual proof.

ARGUMENT: Let $\varepsilon > 0$ be given. Assume that $0 < |x-1| < \frac{\varepsilon}{5}$. Then

$$|(2-5x)-(-3)| = |-5x+5| = |-5||x-1| \overset{|x-1|<\frac{\varepsilon}{5}}{<} 5 \cdot \frac{\varepsilon}{5} = \varepsilon.$$

The proof is complete: We have shown that for any $\varepsilon > 0$, if $0 < |x-1| < \delta = \frac{\varepsilon}{5}$, then $|2-5x)-(-3)| < \varepsilon$.

EXAMPLE 5.7. (One More) Prove that $\lim_{x\to -3}(10x+8) = -22$ using the formal definition of limit.

SOLUTION. SCRAP WORK: Find δ. In this case $a = -3$ and $L = -22$. Be careful of all the negatives. Assume that $\varepsilon > 0$ is given. Then

$$\begin{aligned} |(10x+8)-(-22)| < \varepsilon &\overset{\text{Simplify}}{\iff} |10x+30| < \varepsilon \\ &\overset{\text{Factor}}{\iff} 10|x+3| < \varepsilon \\ &\overset{\text{Rewrite}}{\iff} 10|x-(-3)| < \varepsilon \\ &\overset{\text{Solve}}{\iff} |x-(-3)| < \frac{\varepsilon}{10}. \end{aligned}$$

Notice that we wrote $|x+3|$ as $|x-(-3)|$ so that our inequality would have the correct form: $|x-a| < \delta$ with $\delta = \frac{\varepsilon}{10}$. Do the proof.

ARGUMENT: Given $\varepsilon > 0$. Assume that $0 < |x-(-3)| < \frac{\varepsilon}{10}$. Then

$$|(10x+8)-(-22)| = |10x+30| = 10|x+3| = 10|x-(-3)| < 10 \cdot \frac{\varepsilon}{10} = \varepsilon.$$

Short and sweet!

EXAMPLE 5.8. (Something different) Prove that $\lim_{x\to 3}|18-6x| = 0$ using the formal definition of limit.

SOLUTION. SCRAP WORK: Find δ. In this case $a = 3$ and $L = 0$. Assume that $\varepsilon > 0$ is given. Then

$$\begin{aligned} ||18-6x|-0| < \varepsilon &\overset{\text{Simplify}}{\iff} |18-6x| < \varepsilon \\ &\overset{\text{Factor}}{\iff} |-6||x-3| < \varepsilon \overset{\text{Solve}}{\iff} |x-3| < \frac{\varepsilon}{6}. \end{aligned}$$

So $\delta = \frac{\varepsilon}{6}$. Do the proof.

ARGUMENT: Given $\varepsilon > 0$. Assume that $0 < |x-3| < \frac{\varepsilon}{6}$. Then

$$||18-6x|-0| = |18-6x| = 10|x+3| = |-6||x-3| < 6 \cdot \frac{\varepsilon}{6} = \varepsilon.$$

YOU TRY IT 5.5. Use the formal definition to prove each of the following.

(*a*) $\lim_{x\to 5}(4x+7) = 27$ (*b*) $\lim_{x\to -4}(-2x-17) = -9$ (*c*) $\lim_{x\to 12}\left(\frac{x}{2} - 11\right) = -5$

The Stuff You Need to Turn In Next Class. Name(s):

You should work with one partner if at all possible. If you do, **hand in one sheet for both of you**. With a partner, start up the limit applet at `http://math.hws.edu/~mitchell/Math130F12/Web/LimitsNew.html` or use the link at the course website. Besides the startup example that we did in class, the applet contains three other examples from the text, two from the text. To load one of the examples, select it from the pop-up menu at the top of the applet and click the "Load Example" button. For each value of ε, see if you can find a value of δ such that works: so that for every x with $0 < |x - a| < \delta$, it follows that $|f(x) - L| < \varepsilon$. Report your results (including the numerical value of each δ, if you can find one that works). For each example, carefully state what you can conclude about $\lim_{x \to a} f(x) = L$ and why. *Don't forget that you can click at a point to zoom in on it!*

1. **Homework 1:** $f(x) = 1/(x-1)$. This is very similar to the startup example. Fill in the values of a, L, and $f(a)$, then try to find delta for each of the given epsilon values. Decide whether the limit exists and justify your answer.

$a =$	$L =$		$f(a) =$	
Epsilon ε	0.1	0.05	0.01	0.001
Delta δ				
Conclusion: For each ε does there appear to be a corresponding δ? Does the limit exist?				

2. **Homework 1:** $f(x) = x^2 - 1$. Fill in the values of a, L, and $f(a)$, then try to find delta for each of the given epsilon values. Decide whether the limit exists and justify your answer.

$a =$	$L =$		$f(a) =$	
Epsilon ε	0.2	0.1	0.01	0.001
Delta δ				
Conclusion: For each ε does there appear to be a corresponding δ? Does the limit exist?				

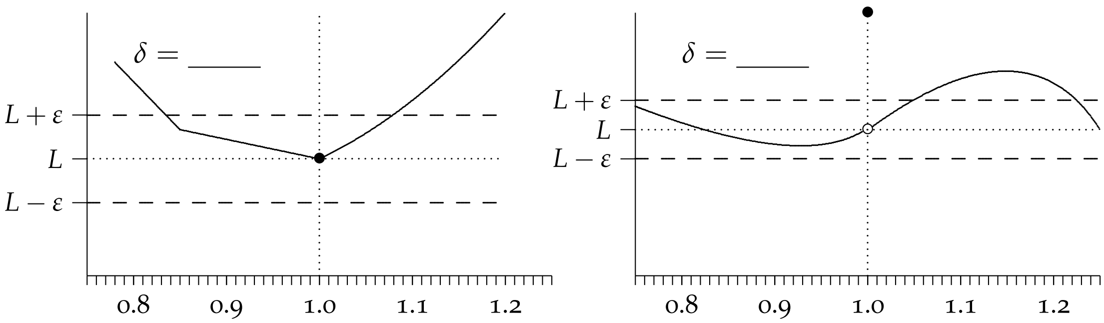
3. **Two Pieces**. Fill in the values of a, L, and $f(a)$, then try to find delta for each of the given epsilon values. Decide whether the limit exists.

$a =$	$L =$		$f(a) =$	
Epsilon ε	0.2	0.1	0.01	0.001
Delta δ				
Conclusion: For each ε does there appear to be a corresponding δ? Does the limit exist?				

4. **Example 3: Look Close!** Fill in the values of a, L, and $f(a)$. Decide whether the limit exists.

$a =$	$L =$		$f(a) =$	
Epsilon ε	.1	.01	.001	.0005
Delta δ				
Conclusion: Does the limit exist? Explain in terms of ε and δ.				

5. In each figure, for the given choice of ε, find and **draw** a δ interval (a vertical strip) about $a = 1$ which satisfies the limit definition. **What is δ in each case?** Note the **scale**. Note: In each figure, the same δ must work on both the left and right at sides of $a = 1$.



6. With your partner use the **formal definition** of limit to prove the following:

$$\lim_{x \to 10} 6x - 7 = 53.$$

Use the same type of careful argument we made in class today with absolute values, ε, and δ.