Trig Limits

Three Important Limits

In this section we will derive some important trig limits. Recall that a few days ago, we used a table of values and a graph to evaluate the following limit:

THEOREM 11.1. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1.$

YOU TRY IT 11.1. The limit above relates to the slope problem. Use the limit above to find the slope of $f(x) = \sin x$ at $x = 0$.

Figure 11.1: As x approaches 0, $\frac{\sin x}{x}$ approaches 1.

    We will give a proof of this a bit later. But for now, let's use this limit to determine some other limits that will turn out to be important in our discussion of the slope problem.

THEOREM 11.2. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$ and $\lim\limits_{x \to 0} \dfrac{\cos(x) - 1}{x} = 0.$

Proof. Notice that this is an indeterminate limit of the form '$\frac{0}{0}$.' We will use Theorem 11.1 by multiplying by a type of conjugate.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} \overset{\text{Conjugate}}{=} \lim_{x \to 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x}$$

$$\overset{\text{Theorem 11.1}}{=} 1 \cdot \frac{0}{2}$$

$$= 0.$$

Finally, notice that $\lim\limits_{x \to 0} \dfrac{\cos(x) - 1}{x} = -\left(\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} \right) \overset{\text{Above}}{=} 0.$ □

THEOREM 11.3. $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1.$

Proof. This, too, is an indeterminate limit of the form '$\frac{0}{0}$.' We will use Theorem 11.1 again.

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \overset{\text{Theorem 11.1}}{=} 1 \cdot \frac{1}{1} = 1.$$

□

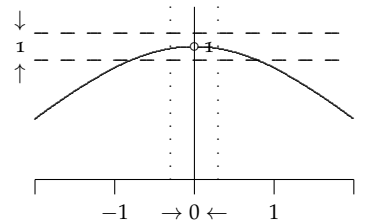The key element in all three limits is that the 'angle' that appears in the numerator is exactly the angle that appears in the denominator. In each of these limits, there is nothing special about the variable x. For example, $\lim\limits_{\circledast \to 0} \dfrac{\sin \circledast}{\circledast} = 1$ or $\lim\limits_{\circledast \to 0} \dfrac{1 - \cos \circledast}{\circledast} = 0$, where $\circledast$ represents some variable quantity is approaching 0.

Here's an example:

$$\lim_{\theta \to 0} \frac{\sin 10\theta}{10\theta} = 1$$

because as $\theta \to 0$ we also have $10\theta \to 0$. Notice again that the 'angle' that appears in the numerator, 10θ, is exactly the angle that appears in the denominator.

Examples and Variations

You should memorize the three limits above. They will be used in a variety of situations.

EXAMPLE 11.1. (Make it so) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 4x}{x}$.

SOLUTION. This is indeterminate, of the form '$\frac{0}{0}$.' The problem is that the 'angle' or 'argument' is not the same in the numerator and denominator. In the words of Captain Picard of the starship *USS Enterprise*, "Make it so!" We want $4x$ in the denominator, so if we multiply the denominator by 4, then we must multiply the numerator by 4. "Let's do it!"[1]

> [1] Gary Gilmore's last words in Norman Mailer's *The Executioner's Song*.

$$\lim_{x \to 0} \frac{\sin 4x}{x} = \lim_{x \to 0} \frac{4 \sin 4x}{4x} = 4 \lim_{x \to 0} \frac{\sin 4x}{4x} \overset{\text{Theorem 11.1}}{=} 4(1) = 4.$$

The limit is 4, not 1.

EXAMPLE 11.2. (Make it so) Evaluate $\lim\limits_{\theta \to 0} \dfrac{\tan 2\theta}{6\theta}$.

SOLUTION. This time we want 2θ in the denominator, so

$$\lim_{\theta \to 0} \frac{\tan 2\theta}{6\theta} = \lim_{\theta \to 0} \frac{\tan 2\theta}{3 \cdot 2\theta} = \frac{1}{3} \cdot \lim_{\theta \to 0} \frac{\tan 2\theta}{2\theta} \overset{\text{Theorem 11.3}}{=} \frac{1}{3}(1) = \frac{1}{3}.$$

EXAMPLE 11.3. (Make it so) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 2x}{\sin 3x}$.

SOLUTION. This, too, is indeterminate, of the form '$\frac{0}{0}$.' To take advantage of Theorem 11.1 we need to divide the numerator and denominator by x. So

$$\lim_{x \to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \to 0} \frac{\frac{\sin 2x}{x}}{\frac{\sin 3x}{x}} = \lim_{x \to 0} \frac{2 \cdot \frac{\sin 2x}{2x}}{3 \cdot \frac{\sin 3x}{3x}} \overset{\text{Theorem 11.1}}{=} \frac{2 \cdot 1}{3 \cdot 1} = \frac{2}{3}.$$

EXAMPLE 11.4. (Make it so) Evaluate $\lim\limits_{x \to 0} \dfrac{\tan^2 3x}{x}$.

SOLUTION. Take advantage of Theorem 11.3 by rewriting.

$$\lim_{x \to 0} \frac{\tan^2 3x}{x} = \lim_{x \to 0} \frac{\tan 3x}{x} \cdot \tan 3x = \lim_{x \to 0} \frac{3 \tan 3x}{3x} \cdot \tan 3x \overset{\text{Theorem 11.1}}{=} 3 \cdot 0 = 0.$$

EXAMPLE 11.5. (Make it so) Evaluate $\lim\limits_{x \to 0} \dfrac{1 - \cos 4x}{x}$.

SOLUTION. Take advantage of Theorem 11.3 by rewriting.

$$\lim_{x \to 0} \frac{1 - \cos 4x}{x} = \lim_{x \to 0} \frac{4(1 - \cos 4x)}{4x} = 4 \lim_{x \to 0} \frac{1 - \cos 4x}{4x} \overset{\text{Theorem 11.2}}{=} 4 \cdot 0 = 0.$$

The Squeeze Theorem

The proof of Theorem 11.1 depends on another useful result that is helpful in calculating certain complicated limits.

THEOREM 11.4. (The Squeeze Theorem) Assume that f, g and h are functions such that $f(x) \leq g(x) \leq h(x)$ for all values of x near a, except perhaps at a itself. If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$. In other words, all three limits are equal.

Proof sketch: Since $f(x) \leq g(x) \leq h(x)$, this means $g(x)$ is trapped between $f(x)$ and $h(x)$. Since both $f(x)$ and $h(x)$ approach L as x approaches a, then $g(x)$ must approach a.

Figure 11.2: As x approaches a, $f(x) \to L$ and $h(x) \to L$, so $g(x) \to L$.

> **EXAMPLE 11.6.** Here's a made up example that illustrates the ideas. Suppose that we have some function $g(x)$ and that we know
>
> $$\cos x \leq g(x) \leq x^2 + 1 \qquad (x \neq 0).$$
>
> Determine $\lim_{x \to 0} g(x)$. In this case we are lucky because the limits of the outer functions in the inequality above are equal. Since $\lim_{x \to 0} \cos x = 1$ and $\lim_{x \to 0} x^2 + 1 = 1$, then by the squeeze theorem the function in the middle has the same limit: $\lim_{x \to 0} g(x) = 1$ also.

> **EXAMPLE 11.7.** Here's another. Suppose $g(x)$ satisfies
>
> $$3x^2 + 1 \leq g(x) \leq (x+1)^2.$$
>
> Determine $\lim_{x \to 1} g(x)$. This time the limits of the outer functions are equal to 4 Since $\lim_{x \to 1} 3x^2 + 1 = 4$ and $\lim_{x \to 1} (x+1) = 4$. So by the squeeze theorem the function in the middle has the same limit: $\lim_{x \to 1} g(x) = 4$.

> **EXAMPLE 11.8.** The function $\sin \frac{1}{x}$ is not defined at 0. We also saw from its graph that $\lim_{x \to 0} \sin \frac{1}{x}$ DNE. Using the graph, does $\sin \frac{1}{x}$ have a VA or an DR at $x = 0$?
>
> Since we are dealing with the sine function, it is still the case that
>
> $$-1 \leq \sin \frac{1}{x} \leq 1 \qquad (x \neq 0).$$
>
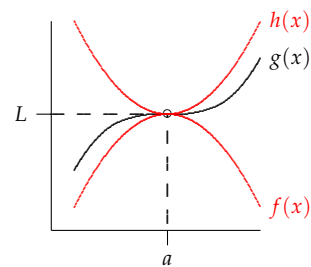> Now consider a related function function $g(x) = x^2 \sin \frac{1}{x}$. Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.
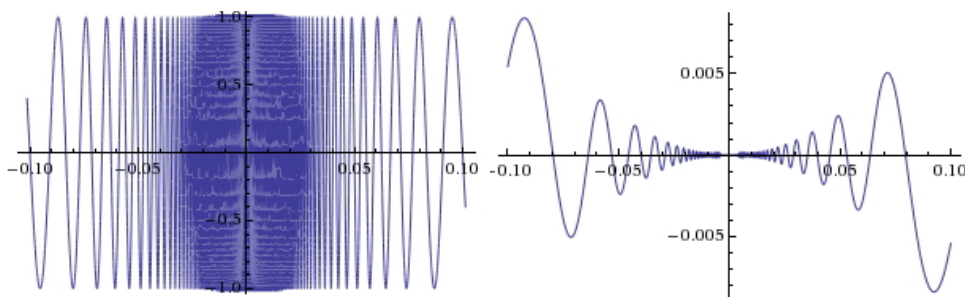

Figure 11.3: Left: The function $\sin \frac{1}{x}$. Right: The function $g(x) = x^2 \sin \frac{1}{x}$.

> **SOLUTION.** There is no factoring or simplification that we can do. It's not even an indeterminate form. But what we can see is that since
>
> $$-1 \leq \sin \frac{1}{x} \leq 1 \qquad (x \neq 0).$$
>
> and since $x^2 \geq 0$, if we multiply through the inequality above by x^2 we get
>
> $$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2 \qquad (x \neq 0).$$

Notice that the two 'outside' functions have the same limit (namely 0) at 0. So let's apply the Squeeze Theorem where $f(x) = -x^2$ and $h(x) = x^2$. Then

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} -x^2 = 0 \quad \text{and} \quad \lim_{x \to 0} h(x) = \lim_{x \to 0} x^2 = 0.$$

So we get to conclude the same about the middle function $g(x)$:

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 \sin \tfrac{1}{x} = 0.$$

The Proof of a Key Trig Limit

To prove $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$, we will use a bit of trig and geometry. In particular we will compare three areas that are determined by a unit circle centered at the origin. To start, look at Figure 11.4.
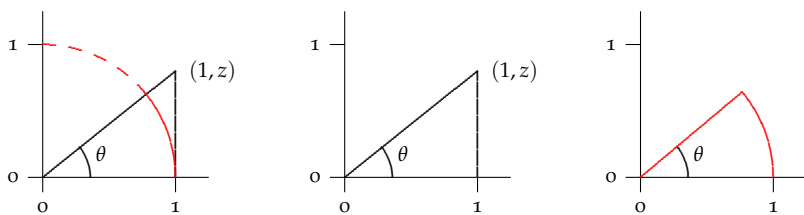

Figure 11.4: The sector area of the circle has a smaller area than the corresponding triangle.

In the large triangle, $\tan \theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{z}{1} = z$. So the triangle has height $z = \tan \theta$ and base 1, so its area is

$$\text{Area(Big Triangle)} = \tfrac{1}{2}(1)(z) = \tfrac{1}{2} \tan \theta.$$

Next, the sector area (see the right side of Figure 11.4) as a fraction of the entire circle. The sector is $\dfrac{\theta}{2\pi}$ of the entire circle, so its area is

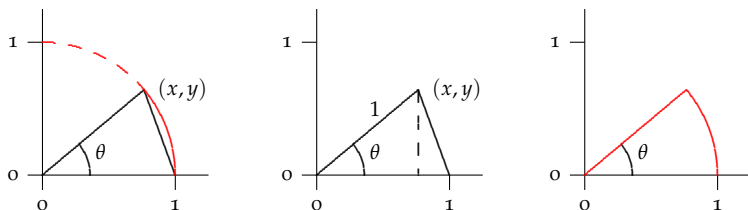$$\text{Area(sector)} = \frac{\theta}{2\pi} \cdot \pi (1)^2 = \frac{\theta}{2}.$$

Figure 11.5: The sector area of the circle has a larger area than the corresponding triangle.

The triangle within the sector has height y. But $y = \tfrac{y}{1} = \tfrac{\text{opp}}{\text{hyp}} = \sin \theta$. So the small triangle has height $y = \sin \theta$ and base 1, so its area is

$$\text{Area(small triangle)} = \tfrac{1}{2}(1)(y) = \tfrac{1}{2} \sin \theta.$$

Now we set up the Squeeze Theorem

$$\text{Area(Big Triangle)} \geq \text{Area(Sector)} \geq \text{Area(small triangle)}$$

Using the equations worked out above, this translates to

$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

and multiplying by 2 and rewriting $\tan\theta$

$$\frac{\sin\theta}{\cos\theta} \geq \theta \geq \sin\theta.$$

Taking reciprocals reverses the inequalities

$$\frac{\cos\theta}{\sin\theta} \leq \frac{1}{\theta} \leq \frac{1}{\sin\theta}.$$

Multiply through by $\sin\theta$ to get

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

Now take the limits as $\theta \to 0$ of the first and last functions,

$$\lim_{\theta\to 0} \cos\theta = 1 \quad \text{and} \quad \lim_{\theta\to 0} 1 = 1.$$

So we can apply the Squeeze Theorem, and we conclude $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ also.

EXAMPLE 11.9 (Review). Find the tangent slope of the curve $y = f(x) = \sin 3x$ at the point $(0,0)$ on the curve by taking the limit of secant slopes. Then find the equation of the tangent line there.

SOLUTION. Remember that the secant slopes of f at $(0,0)$ are given by

$$m_{\text{sec}} = \frac{f(x) - f(0)}{x - 0} = \frac{\sin 3x - \sin 0}{x - 0} = \frac{\sin 3x}{x}.$$

So taking the limit as $x \to 0$ we get

$$m_{\text{tan}} = \lim_{x\to 0} m_{\text{sec}} = \lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{3\sin 3x}{3x} = 3.$$

So the slope of the tangent line is 3 and the point is $(0,0)$. The equation of the line is

$$y - 0 = 3(x - 0) \quad \text{or} \quad y = 3x.$$

EXAMPLE 11.10 (Review). Find the tangent slope of the curve $y = f(x) = \cos x - \tan 2x$ at the point $(0,1)$ on the curve by taking the limit of secant slopes.

SOLUTION. Check that $f(0) = 1$. The secant slopes of f at $(0,1)$ are given by

$$m_{\text{sec}} = \frac{f(x) - f(0)}{x - 0} = \frac{(\cos x - \tan 2x) - (1 - 0)}{x - 0} = \frac{(\cos x - \tan 2x) - 1}{x}.$$

So taking the limit as $x \to 0$ we get

$$\begin{aligned}
m_{\text{tan}} = \lim_{x\to 0} m_{\text{sec}} = \lim_{x\to 0} \frac{(\cos x - \tan 2x) - 1}{x} &= \lim_{x\to 0} \frac{(\cos x - \tan 2x) - 1}{x} \\
&= \lim_{x\to 0} \frac{(\cos x - 1) - \tan 2x}{x} \\
&= \lim_{x\to 0} \frac{\cos x - 1}{x} - \frac{\tan 2x}{x} \\
&= \lim_{x\to 0} \frac{\cos x - 1}{x} - \frac{2\tan 2x}{2x} \\
&= -2.
\end{aligned}$$

The slope of the tangent line is -2 and the point is $(0,1)$. The equation of the line is

$$y - 1 = -2(x - 0) \quad \text{or} \quad y = -2x + 1.$$

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YOU TRY IT 11.2. Practice with trig limits. Show your work!

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{12x}$

(b) $\lim\limits_{x\to 0} \dfrac{\sin(-8x)}{2x}$

(c) $\lim\limits_{x\to 0} \dfrac{\tan(-2x)}{x}$

(d) $\lim\limits_{x\to 0} \dfrac{\tan(5x)}{\sin(3x)}$

(e) $\lim\limits_{x\to 0} \dfrac{1-\cos^2(3x)}{x(1+\cos(3x))}$

(f) $\lim\limits_{x\to 0} \dfrac{2\tan^2 x}{x^2}$

(g) $\lim\limits_{x\to 0} \dfrac{2\tan^2 x}{x}$

(h) $\lim\limits_{x\to \pi/2} \dfrac{\sin x}{x}$

Answers: $\frac{1}{12}$; -4; -2; $\frac{5}{3}$; 0; 2; 0; $\frac{2}{\pi}$.