

## Math 131 Lab 8

☞ Do \* problems first. Then do the others. Brief answers are on the back. Reduction formulas on back

1. **For the Exam: Higher Powers of Trig Functions** Use the rules we developed in class to determine

- a)  $\int \sin^2 2x \cos^3 2x \, dx$       b)  $\int \cos^4 3x \, dx$       c)  $\int \sec^5(2x) \, dx$       d)  $\int \sin^3(5x) \cos^{-7}(5x) \, dx$   
e) Now try  $\int \tan^2(\pi x) \sec^2(\pi x) \, dx$ . Think, first!

2. **Low Powers of Trig Functions** which you should already know.

- a)  $\int \cos 9x \, dx$       b)  $\int \tan^2(\pi x) \, dx$       c)  $\int \sin^2(-7x) \, dx$

3. Before working these out, classify each by the technique that applies: mentally adjust, substitution, parts, parts twice, or ordinary methods. Check your answers by differentiating.

- a)  $\int x e^{x^2} \, dx$       b)  $\int_0^1 (x^2 - x + 1)e^x \, dx$       c)  $\int_0^\pi (x - \pi) \sin x \, dx$   
d)  $\int e^x \sin(e^x) \, dx$       e)  $\int e^{3x} \sin x \, dx$       f)  $\int \sin(5x) \cos(x) \, dx$

4. a) \*Assume that  $n \neq -1$ . Determine a formula for  $\int x^n \ln x \, dx$ .

b) Use your answer to quickly determine (without further integration)  $\int \frac{\ln x}{x^6} \, dx$ .

5. a) \*Determine  $\int (\ln x)^2 \, dx$ .

b) \*Let  $R$  be the region enclosed by  $y = \ln x$ , the  $x$ -axis, and  $x = e$  in the first quadrant. Rotate  $R$  about the  $x$ -axis and find the volume. Use your answer to part (a).

6. **Area Review:** Find the area of the region enclosed by  $y = \arcsin x$ , the  $x$ -axis, and  $x = \sqrt{2}/2$  in the first quadrant by integrating along the  $x$ -axis.

7. a) \***Work Review:** A tank is formed from rotating the region enclosed by  $y = \arcsin(x^2)$ , the  $y$ -axis,  $y = \pi/2$ , and the  $x$ -axis. The tank is full of whale oil (60 lbs/ft<sup>3</sup>). Find the work “lost” in draining the tank through a hole in the bottom

b) Set up the integral for the work is lost if only the top half is drained.

8. \***Volume Review:** Let  $R$  be the region in the first quadrant enclosed by  $y = \cos x$ ,  $y = \sin x$ , and the  $y$ -axis.

a) Rotate  $R$  about the  $x$ -axis and find the volume using the disk method.

b) Rotate  $R$  about the  $y$ -axis and find the volume by using the shell method.

9. **WeBWorK setDay21.** Let  $R$  be the region enclosed by  $y = \ln x$ , the  $y$ -axis, the  $x$ -axis, and  $y = 1$  in the first quadrant. Rotate  $R$  about the  $y$ -axis to form a tank. If liquid has density 60 lbs/cu. ft., how much work is required to fill it through a hole in the bottom of the tank? Hint: During the integration you can use Problem 4.

10. Let  $R$  be the region enclosed by  $y = e^x$ , the  $y$ -axis, and  $y = e$  in the first quadrant. Rotate  $R$  about the  $y$ -axis to form a tank. If it is full of a liquid whose density is 60 lbs/cu. ft., how much work is lost if it leaks out the bottom and drops to ground level?

11. a) In the past we have done the problem  $\int \frac{\ln x}{x} \, dx$  by  $u$ -substitution. Show that **you can do it by parts**—but as a problem that cycles back on itself (a technique that we had not seen until Day 21).

b) Check your answer by doing the same problem via substitution.

12. a) **WeBWorK setExtraCredit2.** Assume that  $n$  is a positive integer. Create a reduction formula for  $\int x^n e^x \, dx$  by using integration by parts once. Hint: Let  $u = x^n$ .

b) Extra Credit: Use your reduction formula (repeatedly) to find  $\int x^3 e^x \, dx$ .

**Math 131 Lab 8 Brief Answers.** Caution: Be careful of typos. **Extra Credit** if you are the first to find one.

1. a)  $\frac{\sin^3 2x}{6} - \frac{\sin^5 2x}{10} + c$
- b)  $\frac{\cos^3 3x \sin 3x}{12} + \frac{\cos 3x \sin 3x}{8} + \frac{3x}{8} + c$
- c)  $\frac{\sec^3(2x) \tan(2x)}{8} + \frac{3 \sec(2x) \tan(2x)}{16} + \frac{3 \ln |\sec(2x) + \tan(2x)|}{16} + c$
- d)  $\frac{\cos^{-6}(5x)}{30} - \frac{\cos^{-4}(5x)}{20} + c$
- e)  $\frac{1}{3\pi} \tan^3(\pi x) + c$

2. Use half-angle for (c).

$$\text{a) } \frac{1}{9} \sin 9x + c \quad \text{b) } \frac{1}{\pi} \tan(\pi x) - x + c \quad \text{c) } \frac{x}{2} + \frac{1}{28} \sin(-14x) + c$$

3. Sketches:

$$\begin{array}{lll} \text{a) } \frac{1}{2}e^{x^2} + c & \text{b) } (x^2 - 3x + 4)e^x \Big|_0^1 = 2e - 4 & \text{c) } -(x - \pi) \cos x + \sin x \Big|_0^\pi = -\pi \\ \text{d) } -\cos e^x + c & \text{e) } \frac{1}{10}(3e^{3x} \sin x - e^{3x} \cos x) + c & \text{f) } -\frac{1}{24}[\sin(5x) \sin x + 5 \cos(5x) \cos x] + c \end{array}$$

4. a)  $\frac{1}{n+1}x^{n+1} \ln x - \frac{1}{(n+1)^2}x^{n+1} + c$ ; b)  $-\frac{1}{5}x^{-5} \ln(x) - \frac{1}{25}x^2 + c$

5. a) Use parts twice:  $x(\ln x)^2 - 2x \ln x + 2x + c$ .

b) Use (a):  $\pi(e - 2)$ .

6. By parts:  $x \arcsin x + \sqrt{1-x^2} \Big|_0^{\sqrt{2}/2} = \frac{\sqrt{2}}{8}\pi + \sqrt{2}/2 - 1$ .

7. a)  $-60\pi$  lbf.

8. a)  $\frac{1}{2}\pi$ . b)  $\frac{\sqrt{2}}{2}\pi^2 - 2\pi$ .

9.  $-15\pi(e^2 + 1)$  ft-lbs.

10.  $-15\pi(e^2 - 1)$  ft-lbs.

11.  $\frac{(\ln x)^2}{2} + c$ .

12. a)  $x^n e^x - n \int x^{n-1} e^x dx$ . b)  $x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c$ .

### Reduction Formulas for Large Powers.

1)  $\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$

2)  $\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$

3)  $\int \tan^n x dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx$

4)  $\int \sec^n x dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx$

## Math 131 Lab 8 Answers

1. a) Since the power of the cosine function is odd, we use Guideline #2: Split off a power of  $\cos 2x$ , then let  $u = \sin 2x$ , so  $\frac{1}{2} du = \cos 2x dx$ .

$$\begin{aligned} \int \sin^2 2x \cos^3 2x dx &= \int \sin^2 2x \cos^2 2x \cdot \cos 2x dx = \int \sin^2 2x (1 - \sin^2 2x) \cdot \cos 2x dx \\ &= \frac{1}{2} \int u^2 (1 - u^2) du = \frac{1}{2} \int u^2 - u^4 du = \frac{1}{2} \left[ \frac{u^3}{3} - \frac{u^5}{5} \right] + c = \frac{\sin^3 2x}{6} - \frac{\sin^5 2x}{10} + c \end{aligned}$$

b) Use reduction formulas. First let  $u = 3x$ , so  $\frac{1}{3} du = dx$ . So  $\int \cos^4 3x dx = \frac{1}{3} \int \cos^4 u du$

$$\begin{aligned} &= \frac{1}{3} \left( \frac{\cos^3 u \sin u}{4} + \frac{3}{4} \int \cos^2 u du \right) = \frac{1}{3} \left( \frac{\cos^3 u \sin u}{4} + \frac{3}{4} \left[ \frac{\cos u \sin u}{2} + \frac{1}{2} \int 1 du \right] \right) \\ &= \frac{1}{3} \left( \frac{\cos^3 u \sin u}{4} + \frac{3}{4} \left[ \frac{\cos u \sin u}{2} + \frac{u}{2} \right] \right) + c = \frac{\cos^3 3x \sin 3x}{12} + \frac{\cos 3x \sin 3x}{8} + \frac{3x}{8} + c \end{aligned}$$

- c) Use reduction formula and substitution:  $u = 2x \Rightarrow \frac{1}{2} du = dx$ . So

$$\begin{aligned} \int \sec^5(2x) dx &= \frac{1}{2} \int \sec^5 u du = \frac{1}{2} \left[ \frac{\sec^3 u \tan u}{4} + \frac{3}{4} \int \sec^3 u du \right] = \frac{1}{2} \left[ \frac{\sec^3 u \tan u}{4} + \frac{3}{4} \left[ \frac{\sec u \tan u}{2} + \frac{1}{2} \int \sec u du \right] \right] \\ &= \frac{1}{2} \left[ \frac{\sec^3 u \tan u}{4} + \frac{3 \sec u \tan u}{8} + \frac{3 \ln |\sec u + \tan u|}{8} \right] + c = \frac{\sec^3(2x) \tan(2x)}{8} + \frac{3 \sec(2x) \tan(2x)}{16} + \frac{3 \ln |\sec(2x) + \tan(2x)|}{16} + c \end{aligned}$$

- d) Use  $u = \cos(5x)$  so  $-\frac{1}{5} du = \sin(5x) dx$ .

$$\begin{aligned} \int \sin^3(5x) \cos^{-7}(5x) dx &= \int \sin^2(5x) \cos^{-7}(5x) \sin(5x) dx = \int [1 - \cos^2(5x)] \cos^{-7}(5x) \sin(5x) dx \\ &= -\frac{1}{5} \int (1 - u^2) u^{-7} du = -\frac{1}{5} \int u^{-7} - u^{-5} du = \frac{u^{-6}}{30} - \frac{u^{-4}}{20} + c = \frac{\cos^{-6}(5x)}{30} - \frac{\cos^{-4}(5x)}{20} + c \end{aligned}$$

- e)  $u$ -substitution.

|               |                            |   |
|---------------|----------------------------|---|
| Substitution: | $u = \tan \pi x$           | $= \frac{1}{\pi} \int u^2 du = \frac{1}{3\pi} u^3 + c = \frac{1}{3\pi} \tan^3(\pi x) + c$ |
|               | $du = \pi \sec^2 \pi x dx$ | $\frac{1}{\pi} du = x dx$   |

2. a) Mental adjustment ( $u = 9x$ ).  $\int \cos 9x dx = \frac{1}{9} \int \cos u du = \frac{1}{9} \sin u + c = \frac{1}{9} \sin 9x + c$ .

b) Trig id:  $\int \tan^2(\pi x) dx = \int \sec^2(\pi x) - 1 dx = \frac{1}{\pi} \tan(\pi x) - x + c$ .

- c) Use half-angle formula and followed by a mental adjustment:

$$\int \sin^2(7x) dx = \int \frac{1}{2} - \frac{1}{2} \cos(-14x) dx = \frac{1}{2} x + \frac{1}{28} \sin(-14x) + c$$

3. a) Substitution ( $u = x^2 \Rightarrow du = 2x dx$ ).  $\Rightarrow \frac{1}{2} \int e^u du = \frac{1}{2} e^u + c = \frac{1}{2} e^{x^2} + c$ .

- b) Parts twice:

|                   |               |   |
|-------------------|---------------|---|
| $u = x^2 - x + 1$ | $dv = e^x dx$ | $= (x^2 - x + 1)e^x \Big _0^1 - \int_0^1 (2x - 1)e^x dx$                    |
| $du = (2x - 1)dx$ | $v = e^x$     |   |
| $u = 2x - 1$      | $dv = e^x dx$ | $= (x^2 - x + 1)e^x \Big _0^1 - [(2x - 1)e^x \Big _0^1 - \int_0^1 2e^x dx]$ |
| $du = 2dx$        | $v = e^x$     | $= (x^2 - x + 1)e^x - (2x - 1)e^x + 2e^x \Big _0^1 = 2e - 4$                |

- c) Parts:

|               |                  |   |
|---------------|------------------|---|
| $u = x - \pi$ | $dv = \sin x dx$ | $= -(x - \pi) \cos x \Big _0^\pi + \int_0^\pi \cos x dx$            |
| $du = dx$     | $v = -\cos x$    | $= -(x - \pi) \cos x + \sin x \Big _0^\pi = (0 + 0) - (\pi) = -\pi$ |

- d) Substitution ( $u = e^x \Rightarrow du = e^x dx$ ).  $\Rightarrow \int \sin u du = -\cos u + c = -\cos(e^x) + c$ .

e) Parts twice, "circle around". Note signs and constants:

|                                    |                                   |   |
|------------------------------------|-----------------------------------|---|
| $u = e^{3x}$<br>$du = 3e^{3x} dx$  | $dv = \sin x dx$<br>$v = -\cos x$ | $\int e^{3x} \sin x dx = -e^{3x} \cos x + \int 3e^{3x} \cos x dx$   |
| $u = 3e^{3x}$<br>$du = 9e^{3x} dx$ | $dv = \cos x dx$<br>$v = \sin x$  | $\int e^{3x} \sin x dx = -e^{3x} \cos x + 3e^{3x} \sin x - 9 \int e^{3x} \sin x dx$<br>So, $10 \int e^{3x} \sin x dx = e^{3x}(3 \sin x - \cos x) + c$<br>Thus, $\int e^{3x} \sin x dx = \frac{1}{10} e^{3x}(3 \sin x - \cos x) + c$ |

f) Parts twice, "circle around". Note signs and constants:

|  |                                   |  |
|--|-----------------------------------|--|
| $u = \sin(5x)$<br>$du = 5 \cos(5x) dx$     | $dv = \cos x dx$<br>$v = \sin x$  | $\int \sin(5x) \cos x dx = \sin(5x) \sin x - \int 5 \cos(5x) \sin x dx$  |
| $u = 5 \cos(5x)$<br>$du = -25 \sin(5x) dx$ | $dv = \sin x dx$<br>$v = -\cos x$ | $\int \sin(5x) \cos x dx = \sin(5x) \sin x + 5 \cos(5x) \cos x + \int 25 \sin(5x) \cos x dx$<br>So, $-24 \int \sin(5x) \cos x dx = \sin(5x) \sin x + 5 \cos(5x) \cos x + c$<br>Thus, $\int \sin(5x) \cos x dx = -\frac{1}{24} [\sin(5x) \sin x + 5 \cos(5x) \cos x] + c$ |

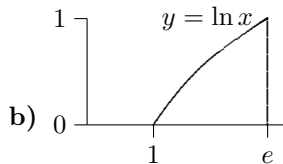
4. a) Parts: 

|                                 |  |  |
|---------------------------------|--|--|
| $u = \ln x$<br>$du = x^{-1} dx$ | $dv = x^n dx$<br>$v = \frac{1}{n+1} x^{n+1}$ | $\int x^n \ln x dx = \frac{1}{n+1} x^{n+1} \ln x - \int \frac{1}{n+1} x^n dx$<br>$\int x^n \ln x dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + c$ |
|---------------------------------|--|--|

b)  $-\frac{1}{5} x^{-5} (\ln(x) + \frac{1}{5}) + c.$

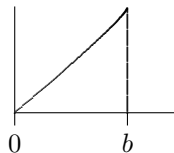
5. a) Parts twice: 

|  |                      |   |
|--|----------------------|---|
| $u = (\ln x)^2$<br>$du = \frac{2 \ln x}{x} dx$ | $dv = dx$<br>$v = x$ | $= x(\ln x)^2 - 2 \int \ln x dx.$ Use parts again in problem #4 with $n = 0.$<br>$= x(\ln x)^2 - 2x \ln x + 2x + c$ |
|--|----------------------|---|



b) From (a):  $\int_1^e \pi (\ln x)^2 dx = \pi [x(\ln x)^2 - 2x \ln x + 2x] \Big|_1^e = \pi(e - 2).$

6.  $A = \int_0^{b=\sqrt{2}/2} \arcsin x dx$



Parts:

|   |   |   |
|---|---|---|
| $u = \arcsin x$<br>$du = \frac{1}{\sqrt{1-x^2}} dx$ | $dv = dx$<br>$v = x$                      | $= x \arcsin x \Big _0^{\sqrt{2}/2} - \int_0^{\sqrt{2}/2} \frac{x}{\sqrt{1-x^2}} dx$  |
| Substitution:<br>$du = -2x dx$                      | $u = 1 - x^2$<br>$-\frac{1}{2} du = x dx$ | $= x \arcsin x \Big _0^{\sqrt{2}/2} + \frac{1}{2} \int u^{-1/2} du = x \arcsin x \Big _0^{\sqrt{2}/2} + u^{1/2}$<br>$= x \arcsin x + \sqrt{1-x^2} \Big _0^{\sqrt{2}/2} = \frac{\sqrt{2}}{8} \pi + \frac{\sqrt{2}}{2} - 1$ |

7. a) Save work! Since the radius of a cross-section of the tank is  $x$ , we need to solve for  $x^2$  (not  $x$ ):  $y = \arcsin(x^2)$ , so

$x^2 = \sin y.$   $W = 60 \int_0^{\pi/2} \pi(0 - y)(\sin y) dy = -60\pi \int_0^{\pi/2} y \sin y dy.$  Use parts:

|                      |                                   |   |
|----------------------|-----------------------------------|---|
| $u = y$<br>$du = dy$ | $dv = \sin y dy$<br>$v = -\cos y$ | $= -60\pi \left[ -y \cos y \Big _0^{\pi/2} + \int_0^{\pi/2} \cos y dy \right]$<br>$= -60\pi \left[ (0 - 0) + \sin y \Big _0^{\pi/2} \right] = (0 + 0) + (1 - 0) = -60\pi \text{ lbf}$ |
|----------------------|-----------------------------------|---|

b) Only the lower limit changes to  $\pi/4$  since the upper part of the tank drains out:  $W = 60 \int_{\pi/4}^{\pi/2} \pi(0 - y)(\sin y) dy$

8. a)  $V = \pi \int_0^{\pi/4} \cos^2 x - \sin^2 x \, dx = \pi \int_0^{\pi/4} (\frac{1}{2} + \frac{1}{2} \cos 2x) - (\frac{1}{2} - \frac{1}{2} \cos 2x) \, dx = \pi \int_0^{\pi/4} \cos 2x \, dx = \frac{1}{2} \pi \sin(2x) \Big|_0^{\pi/4} = \frac{1}{2} \pi.$

b)  $V = 2\pi \int_0^{\pi/4} x \cos x - x \sin x \, dx = 2\pi \int_0^{\pi/4} x(\cos x - \sin x) \, dx.$  Use parts

|  |   |
|--|---|
| $u = x$<br>$dv = (\cos x - \sin x) dx$<br>$du = dx$<br>$v = \sin x + \cos x$ | $2\pi \int_0^{\pi/4} x(\cos x - \sin x) \, dx = 2\pi [x(\sin x + \cos x) - \int (\sin x + \cos x) \, dx]_0^{\pi/4}$                             |
|  | $2\pi \int_0^{\pi/4} x \cos x - x \sin x \, dx = 2\pi [x(\sin x + \cos x) + \cos x - \sin x \, dx]_0^{\pi/4} = \frac{\sqrt{2}}{2} \pi^2 - 2\pi$ |

So  $\pi \int_1^{e^2} (\ln x)^2 \, dx = \pi [x(\ln x)^2 - 2x \ln x + 2x]_1^{e^2} = \pi [(4e^2 - 4e^2 + 2e^2) - (0 - 0 + 1)] = \pi(2e^2 - 1).$

9. Since  $y = \ln x$ , then  $x = e^y$ . So  $W = \int_0^1 60\pi(e^y)^2(y - 0) \, dy = 60\pi \int_0^1 y e^{2y} \, dy.$  By parts

|   |   |
|---|---|
| $u = y$<br>$dv = e^{2y} \, dy$<br>$du = dy$<br>$v = \frac{1}{2} e^{2y}$ | $= 60\pi [\frac{1}{2} y e^{2y} \Big _0^1 - \int_0^1 \frac{1}{2} e^{2y} \, dy]$<br>$= 60\pi [\frac{1}{2} y e^{2y} - \frac{1}{4} e^{2y}]_0^1$ |
|   | $-60\pi [(\frac{1}{2} e^2 - \frac{1}{4} e^2) - (0 - \frac{1}{4})] = -60\pi [\frac{1}{4} e^2 + \frac{1}{4}] = -15\pi(e^2 + 1)$               |

10. Since  $y = e^x$ , then  $x = \ln y$ . So  $W = \int_1^e 60\pi(\ln y)^2(0 - y) \, dy = -60\pi \int_1^e y(\ln y)^2 \, dy.$  By parts twice, the second time using #4 with  $n = 1$ :

|  |  |
|--|--|
| $u = (\ln y)^2$<br>$dv = y \, dy$<br>$du = \frac{2 \ln y}{y} \, dy$<br>$v = \frac{1}{2} y^2$ | $= -60\pi [y(\ln y)^2 \Big _1^e - \int_1^e y \ln y \, dy]$   |
| Use #4 with $n = 1$ :  | $= -60\pi [\frac{1}{2} y^2 (\ln y)^2 - \frac{1}{2} y^2 \ln y + \frac{1}{4} y^2]_1^e$<br>$= -60\pi [(\frac{1}{2} e^2 - \frac{1}{2} e^2 + \frac{1}{4} e^2) - (0 + 0 + \frac{1}{4})] = -15\pi(e^2 - 1)$ |

11. a) Parts:

|   |   |
|---|---|
| $u = \ln x$<br>$dv = \frac{1}{x} \, dx$<br>$du = \frac{1}{x} \, dx$ | $\int \frac{\ln x}{x} \, dx = (\ln x)^2 - \int \frac{\ln x}{x} \, dx$ Cycle!<br>So $2 \int \frac{\ln x}{x} \, dx = (\ln x)^2$ or $\int \frac{\ln x}{x} \, dx = \frac{(\ln x)^2}{2} + c$ |
|---|---|

b) EZ  $u$ -substitution:  $u = \ln x$ ,  $du = \frac{1}{x} \, dx$ . So  $\int \frac{\ln x}{x} \, dx = \int u \, du = \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$

12. a) Parts:

|   |   |
|---|---|
| $u = x^n$<br>$dv = e^x \, dx$<br>$du = nx^{n-1} \, dx$<br>$v = e^x$ | $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$ |
|---|---|

b)  $\int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx = x^3 e^x - 3 \left[ x^2 e^x - 2 \int x e^x \, dx \right]$   
 $= x^3 e^x - 3 \left[ x^2 e^x - 2(xe^x - \int e^x \, dx) \right] = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c.$