A Very Interesting Example

One of the first power series we examined was

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

In Example 15.6 we used the ratio test to show that the interval of convergence was $(-\infty, \infty)$. Since the series converges for x, this means that we can think of this particular power series as a function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

whose domain is all real numbers.

There turns out to be something special about this function. Suppose we try to take the derivative of $f(x)$. We get

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x^n}{n!}\right) \\ &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \cdots .\right) \\ &= \frac{d}{dx}\left(1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \cdots .\right) \\ &= \frac{d}{dx}\left(1 + \frac{2x}{2 \cdot 1} + \frac{3x^2}{3 \cdot 2 \cdot 1} + \frac{4x^3}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{5x^4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \cdots .\right) \\ &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .\right) \end{aligned}$$

Whoa! Notice that $f'(x) = f(x)$. What a minute. The only function we know[1] with this property is $f(x) = e^x$. Amazing! We have just shown that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This means the series could be used to approximate values of e^x. For example,

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

While we might not be able to sum an infinite number of terms, we could compute the first 10 terms to get an approximation

$$e^2 \approx \sum_{n=0}^{10} \frac{2^n}{n!} = 7.38\overline{899470}.$$

A calculator gives the answer as $e^2 = 7.3890560989$. So our approximation was quite close, within about $\frac{6}{100{,}000}$. Of course, the calculator value is only an approximation, too, since e^2 is irrational and has a non-repeating decimal expansion.

So why bother with the series approximation if we have a calculator (laptop, smartphone) available? Well, how do you think the calculator computes e^2? Since the value of e is unknown (exactly) how can the calculator take its square. Well, it does a series approximation (with more terms that we used) to get a very accurate estimate.

[1] Actually, the only function with $f'(x) = f(x)$ is $f(x) = ce^x$, where a is a constant. But in this case, $f(0) = ce^0 = c$. In our particular case where $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, $f(0) = 1$, so $a = 1$, or

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x.$$

Functions from Geometric and Power Series

We can write other functions as power series if we are a bit clever. Remember from our early work with series with geometric series that if $|r| < 1$, then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}. \tag{15.1}$$

Replacing r with x and reading (15.1) backwards we see that we can write the rational function $f(x) = \frac{a}{1-x}$ as a power series:

$$f(x) = \frac{a}{1-x} = \sum_{n=0}^{\infty} ax^n \text{ on } (-1,1). \tag{15.2}$$

Note that a purely geometric series does not converge at the endpoints of the interval ($|r| < 1$). Using this idea and adjusting a and x we can write other functions as power series using a type of "r-substitution" where we are always trying to write the function in the form $\frac{a}{1-r}$.

EXAMPLE 15.11. Write $f(x) = \dfrac{3}{6-2x}$ as a power series. Find its interval of convergence.

SOLUTION. SCRAP WORK: We want to make $f(x) = \frac{3}{6-2x}$ look like $\frac{a}{1-r}$. This requires 'making' the constant 6 in the denominator into a 1. We can do this by dividing the numerator and the denominator by 6.
ARGUMENT: Rewriting we find

$$f(x) = \frac{3}{6-2x} = \frac{\frac{3}{6}}{\frac{6-2x}{6}} = \frac{\frac{1}{2}}{1-\frac{x}{3}}.$$

Now if we think of $a = \frac{1}{2}$ and $r = \frac{x}{3}$, then we have a geometric series:

$$f(x) = \frac{3}{6-2x} = \frac{\frac{1}{2}}{1-\frac{x}{3}} = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{3}\right)^n.$$

To determine the interval of convergence, note that since this is a geometric series we need $|r| < 1$, that is,

$$|r| = \left|\frac{x}{3}\right| < 1 \iff |x| < 3.$$

The radius of convergence is $R = 3$ and the interval of convergence is $(-3,3)$. (Remember that a geometric series does not converge at its endpoints.)

Now here's the deal about Example 15.11: We are saying that $f(x) = \frac{3}{6-2x}$ actually equals (is the same as) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{3}\right)^n$ everywhere in the interval $(-3,3)$. We can either compute the value of f by plugging into the function formula or the series formula—and we will get the same value as long as x is in the interval $9-3,3)$. Suppose we let $x = 2$. Then using the rational function formula,

$$f(2) = \frac{3}{6-2(2)} = \frac{3}{2}.$$

Now using the series formula, we get the same thing because

$$f(2) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n = \frac{\frac{1}{2}}{1-\frac{2}{3}} = \frac{3}{2}. \tag{15.3}$$

What's the point?

Ok, maybe (15.3) is a silly way to compute $f(x) = \frac{3}{6-2x}$. But what if we could express functions such as e^x and $\sin x$ as power series? We don't have a 'formula' to compute values $\sin x$ so having a power series might be helpful. We could evaluate e^x or $\sin x$ by computing the value of the corresponding series. But again, why bother? Just reach over and grab your calculator and do the corresponding calculation. Well, how do you think your calculator does it? It uses series! We'll say more about this shortly. In the meantime, here's a few more examples of converting functions to power series.

Further Examples

EXAMPLE 15.12. Write $f(x) = \dfrac{-24}{8+12x}$ as a power series. Find its interval of convergence.

SOLUTION. We want to make $f(x) = \frac{-24}{8+12x}$ look like $\frac{a}{1-r}$. This time we start by dividing the numerator and denominator by 8 to get a 1 in the denominator.

$$f(x) = \frac{-24}{8+12x} = \frac{-3}{1+\frac{3x}{2}}.$$

Let $a = -3$ and $r = \frac{-3x}{2}$, then we have a geometric series:

$$f(x) = \frac{-24}{8+12x} = \frac{-3}{1+\frac{3x}{2}} = \sum_{n=0}^{\infty} -3\left(-\frac{3x}{2}\right)^n.$$

Since this is a geometric series, the radius of convergence is given by

$$|R| = \left|-\frac{3x}{2}\right| < 1 \iff |x| < \frac{2}{3}.$$

The radius of convergence is $R = \frac{2}{3}$ and the interval of convergence is $(-\frac{2}{3}, \frac{2}{3})$.

EXAMPLE 15.13. Write $f(x) = \dfrac{6}{2+x}$ as a power series centered at $a = 1$. Find its interval of convergence.

SOLUTION. Ok, this time we want to express the series in powers of $x-1$. So we need to rewrite the original function in terms of $x-1$. This is done by replacing x by $x-1$ and making an adjustment and then continuing as in the two previous examples.

$$f(x) = \frac{6}{2+x} = \frac{6}{3+(x-1)} = \frac{2}{1+\frac{(x-1)}{3}} = \sum_{n=0}^{\infty} 2\left(-\frac{(x-1)}{3}\right)^n.$$

Since this is a geometric series, the radius of convergence is given by

$$|R| = \left|-\frac{(x-1)}{3}\right| < 1 \iff |x-1| < 3.$$

The radius of convergence is $R = 3$ and the center is $a = 1$ so the interval of convergence is $(-2, 4)$.

Extending Ideas

Before we do another example, we state an important result about differentiating and integrating power series.

THEOREM 15.2. Let $f(x)$ be the function defined by the power series $\sum c_k(x-a)^k$ on its interval of convergence I.

1. f is continuous on I.
2. f is differentiable on I (except perhaps at it endpoints) and derivative $f'(x) = \sum kc_k(x-a)^{k-1}$ is calculated term-by-term.
3. f is integrable on I (except perhaps at it endpoints) and $f(x)\,dx = \sum \frac{c_k}{k}(x-a)^{k+1}$ is calculated term-by-term.

The next example is now pretty straightforward. But once completed, we will extend it in a couple of important ways.

EXAMPLE 15.14. Write $f(x) = \dfrac{1}{x}$ as a power series centered at $a = 1$. Find its interval of convergence.

SOLUTION. Again we want to express the series in powers of $x - 1$. So we rewrite the original function in terms of $x - 1$ by replacing x by $x - 1$ and making an adjustment.

$$f(x) = \frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty}(-(x-1))^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n.$$

The radius of convergence is given by

$$|R| = |-(x-1)| < 1 \iff |x-1| < 1.$$

The radius of convergence is $R = 1$ and the center is $a = 1$ so the interval of convergence is $(0, 2)$.

Now we since $f(x) = \dfrac{1}{x} = \sum_{n=0}^{\infty}(-(x-1))^n$, by Theorem 15.2 we can integrate $f(x)$ by integrating the series term by term. That is,

$$\int f(x)\,dx = \int \frac{1}{x}\,dx = \int \left(\sum_{n=0}^{\infty}(-1)^n(x-1)^n\right)dx = \sum_{n=0}^{\infty}\int(-1)^n(x-1)^n\,dx. \tag{15.4}$$

Here you can see the meaning of Theorem 15.2: we are able to switch the order of the sum and integration processes. (In general, the order or operations matters, but not in this case.) Since (15.4) is valid, then we know the $\int \frac{1}{x}\,dx = \ln|x| + c$. On the other hand we also know that $\int(-1)^n(x-1)^n\,dx = \dfrac{(-1)^n}{n+1}(x-1)^{n+1} + c$. So we can rewrite (15.4) as

$$\ln x = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(x-1)^{n+1} \qquad 0 < x < 2 \tag{15.5}$$

where C is the constant of integration. By letting $x = 1$ we can evaluate C:

$$\ln 1 = 0 = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(1-1)^{n+1} = C,$$

so $C = 0$. This means that on the interval $(0, 2)$

$$\ln x = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(x-1)^{n+1} = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}\cdots \tag{15.6}$$

Very neat! (Check: Compute $p_\infty(x)$ for $\ln x$ centered at $a = 1$ and see what you get.)

Now here's how this might be used. Suppose you wanted to calculate $\ln 1.1$ Well from (15.6)

$$\ln 1.1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(1.1-1)^{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(0.1)^{n+1}.$$

If we compute just the first four terms of the series, we get an approximation

$$\ln 1.1 \approx \frac{(0.1)}{1} - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = 0.1 - 0.005 + 0.000\overline{3} - 0.000025 = 0.095308\overline{3}.$$

If you take out a calculator you see that the 'actual' value is 0.0953101798. We have made an extremely good approximation after using only a few terms of the series. This is exactly what a calculator does. It does not compute the entire sum, it only computes the first several terms to get an extremely accurate estimation.

EXAMPLE 15.15. Do something clever to find the power series for $f(x) = \arctan x$ centered at $a = 0$. Find the interval of convergence.

SOLUTION. Here's the clever idea. Since $f(x) = \arctan x$, then $f'(x) = \frac{1}{1+x^2}$. We can write the derivative as a power series (think of r as $-x^2$ and a as 1). Remember

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x)^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

The radius of convergence is given by

$$|R| = |x^2| < 1 \iff |x| < 1.$$

The radius of convergence is $R = 1$. Using the alternating series test it is easy to check that the series also converges at $x = \pm 1$ so that interval of convergence is $[-1, 1]$.

Now we do our integration 'magic'

$$\begin{aligned}\arctan x = \int \frac{1}{1+x^2}\,dx &= \int \left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right) dx \\ &= \sum_{n=0}^{\infty}\left(\int (-1)^n x^{2n}\right) dx \\ &= C + \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1},\end{aligned}$$

where we have combined all the constants of integration into a single constant a. By letting $x = 0$ we can evaluate a:.

$$\arctan 0 = 0 = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(0)^{2n+1} = C,$$

so $C = 0$. Assuming this is all legit, this means that on the interval $[-1, 1]$

$$\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\cdots \tag{15.7}$$

Neat-o!

In particular we can evaluate $\frac{\pi}{4} = \arctan 1$ using this series:

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}1^{2n+1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Wow! So now we have a way to approximate π. Using the first 21 terms

$$\pi = 4\cdot\frac{\pi}{4} = 4\arctan 1 \approx 4\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{41}\right) \approx 3.189184782.$$

We see that we need lots more terms to get a better approximation of π.

This is not useful because it takes too many terms to get any accuracy, but there are some related formulas that are very useful. The most famous of this is Machin's formula:

$$\pi/4 = 4\arctan(1/5) - \arctan(1/239)$$

This formula and similar ones were used to push the accuracy of approximations to π to over 500 decimal places by the early eighteenth century (this was all hand calculation!) `http://mathforum.org/library/drmath/view/52543.html`

Great stuff! Do more math!

Postludes

Here are a couple of more series that you might find interesting. It turns out that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

This is already pretty neat. But if we differentiate $\sin x$ we get $\cos x$. Since

$$\frac{d}{dx}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) = \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} = \frac{(-1)^n x^{2n}}{(2n)!} \tag{15.8}$$

we obtain

$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right).$$

In other words, using (15.8),

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Pretty neat stuff! This and more can be found in the last few sections of Chapter 9 in your text.