Math 331: Composition and Integrability

Theorem (Composition and Integrability). Suppose that f is integrable on $[a, b]$ and that $c \leq f(x) \leq d$ for all $x \in [a, b]$. Assume further that g is continuous on $[c, d]$. Then the composite $g \circ f$ is integrable on $[a, b]$.

Examples

Proof: Why would this proof be easy if both f and g were continuous? The proof is a bit complicated notationally. We will use Theorem 3.4.9, so let $\epsilon > 0$. (Review the Sup Lemma before continuing.)

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c) So there is a $\delta' > 0$ so that whenever $s, t \in [c,d]$ and $|s-t| < \delta'$, then

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c) So there is a $\delta' > 0$ so that whenever $s, t \in [c, d]$ and $|s - t| < \delta'$, then $|g(s) - g(t)| < \epsilon$.

[And for technical reasons, we will want to choose $\delta < \epsilon'$. So let $\delta = \min\{\delta', \epsilon'\}$.]

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d) Next, there exists a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ so that $U(P,f) - L(P,f) < \delta^2$ by Theorem 3.4.9 (Fact 5).

e) Now we will show that

$$U(P, g\circ f) - L(P, g\circ f) < \sum_{i=1}^{n}[M_i(g\circ f) - m_i(g\circ f)](x_i - x_{i-1}) < \epsilon.$$

To do this, we separate the set of indices of the partition P into two disjoint sets. On the first set we make $M_i(g\circ f) - m_i(g\circ f)$ small and on the second set we make $\sum(x_i - x_{i-1})$ small. Let

$$A = \{i : M_i(f) - m_i(f) < \delta\} \qquad \text{and} \qquad B = \{i : M_i(f) - m_i(f) \geq \delta\}.$$

If $i \in A$ and $x, y \in [x_{i-1}, x_i]$, then explain why:

$$|f(x) - f(y)| \leq M_i(f) - m_i(f) < \delta.$$

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Sup Lemma.

f) So if $x, y \in [x_{i-1}, x_i]$, then
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h) Adding we get (justify the three inequalities)

$$\begin{aligned}\sum_{i \in A} [M_i(g \circ f) - m_i(g \circ f)](x_i - x_{i-1}) &\leq \sum_{i \in A} \epsilon'(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} \epsilon'(x_i - x_{i-1}) \\ &\leq \epsilon'(b - a)\end{aligned}$$

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So (justify each inequality)

$$\begin{aligned}\sum_{i \in B} 1(x_i - x_{i-1}) &\leq \sum_{i \in B} \left(\frac{M_i(f) - m_i(f)}{\delta}\right)(x_i - x_{i-1}) \\ &\leq \sum_{\text{all } i} \left(\frac{M_i(f) - m_i(f)}{\delta}\right)(x_i - x_{i-1}) \\ &= \frac{U(P,f) - L(P,f)}{\delta} < \delta \leq \epsilon'.\end{aligned}$$

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$$\begin{aligned}
\sum_{i \in B} 1(x_i - x_{i-1}) &\leq \sum_{i \in B} \left(\frac{M_i(f) - m_i(f)}{\delta} \right) (x_i - x_{i-1}) \\
&\leq \sum_{\text{all } i} \left(\frac{M_i(f) - m_i(f)}{\delta} \right) (x_i - x_{i-1}) \\
&= \frac{U(P,f) - L(P,f)}{\delta} < \delta \leq \epsilon'.
\end{aligned}$$

Step (i) above, adding more non-negative terms, Definition of $U(P)$ and $L(P)$.

j) By part (a):

$$\begin{aligned} M_i(g \circ f) - m_i(g \circ f) &\leq \max\{g(t) : t \in [c,d]\} - \min\{g(t) : t \in [c,d]\} \\ &= K \end{aligned}$$

so (justify)

$$\begin{aligned} \sum_{i \in B} [M_i(g \circ f) - m_i(g \circ f)](x_i - x_{i-1}) &\leq \sum_{i \in B} K(x_i - x_{i-1}) \\ &= K \sum_{i \in B} 1(x_i - x_{i-1}) < K\epsilon'. \end{aligned}$$

j) By part (a):

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\end{aligned}$$

Use Step (h) above to justify the last step.

j) Now recombine all the indices:

$$\begin{aligned} U(P, g \circ f) - L(P, g \circ f) &= \sum_{i \in A} [M_i(g \circ f) - m_i(g \circ f)](x_i - x_{i-1}) \\ &\quad + \sum_{i \in B} [M_i(g \circ f) - m_i(g \circ f)](x_i - x_{i-1}) \\ &\leq \epsilon'(b-a) + K\epsilon' \\ &= \end{aligned}$$

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$$= \epsilon'(b - a + K) < \frac{\epsilon}{b - a + K}(b - a + K) = \epsilon$$

So $g \circ f$ is integrable by Theorem 3.4.9.