Math 331 Homework: Day 1

1. a) Review the syllabus. Make sure your understand how you will be assessed.

b) Read: Chapter 1 pages 1–15. Key terms in this reading: **commensurable** and **incommensurable**, **rational** and **irrational**, and **prime**. You should be able to define each term.

c) ☞ Problems to look at (keep a journal): Page 10 #1, 3, 4, 9 10.

d) Look ahead to the next few pages and read the definitions of Dedekind cut and least upper bound.

e) Associated reading for pleasure for the next few classes in Berlinski: Chapters 1–6. See especially his funny proof that $\sqrt{2}$ is irrational.

2. Review material from Math 135 text on the following:

a) proofs by contradiction, uniqueness proofs, and existence proofs;

b) ordered pairs, relations, partial orders, upper bounds, and least upper bounds which we will use in the next few classes.

Homework: Due Next Class

1. Prove that $\sqrt[3]{2}$ is irrational. You will need to prove an intermediate result. (Mimic the proof given in the text that $\sqrt{2}$ is irrational.)

2. If a is irrational, prove that $\sqrt{a}$ is also irrational. What type of proof makes sense?

3. Draw a line segment and call its entire length n. Divide the segment into two (unequal) pieces. The longer part m is chosen so that *the whole is to the longer part as the longer part is to the shorter part.* This ratio of n to m is called ϕ (phi), or the **golden ratio**. Prove that ϕ is irrational. Hint: Use a proof by contradiction. The whole is n, the longer part m, and the shorter part is $n - m$. The statement becomes n is to m as m is to $n - m$, or, algebraically $\phi = \frac{n}{m} = \frac{m}{n-m}$. If ϕ is rational, we may assume $n, m \in \mathbf{N}$ and is in lowest terms. What's the contradiction? (You should be able to do this without determining the actual value of ϕ.)

4. Complete the **Exercises** in the *Close Reading Guide* on page 3 of this assignment to help you understand page 12 in the text. We will not cover this material in great detail.

☞ Class, Practice, and Journal Work

1. EZ: We used: If n is odd, then n^2 is odd. Prove it.

2. We just proved that $\sqrt{2}$ is irrational. We know that $\sqrt{4} = 2$ is rational. But suppose we tried to show $\sqrt{4}$ is irrational by using the same sort of proof that we used to show $\sqrt{2}$ is irrational. The proof must fail some place. Where? What goes wrong with the proof? Understanding where the proof goes wrong will help you to see in what other cases the proof can be used successfully.

3. In high school you learned about decimal expressions for rational numbers. The important facts are that any *repeating decimal* can be expressed as the quotient of two integers, and any quotient of integers can be expressed repeating decimal. (Note: Decimals that *terminate* such as 0.25 should be thought of as repeating with infinitely many 0's, e.g., $0.25\overline{0}$.) Represent each of these repeating decimals as a rational number.

a) 0.3 **b)** $0.\overline{8}$ **c)** $0.\overline{64}$ **d)** $.3\overline{21}$ **e)** $.\overline{a_1 a_2 a_3}$ **f)** $0.\overline{9}$

4. Carefully explain why the decimal expansion of any rational number p/q must repeat. To make it easier, consider only the case where $0 < p < q$. (First, how many possible remainders can occur when doing the long division for p/q. For example, what remainders occur in the division process for the decimal expansion of $3/7$? Next, consider what must happen in the division process when some remainder recurs for the second time. For example, what do you know must happen when the remainder 4 crops up for the second time in $3/7$?)

5. The equivalence between repeating decimals and rationals gives us a way to generate decimal expressions for irrational numbers. All we have to do is invent a pattern that is sure never to repeat. Invent several irrational numbers using this idea.

Math 331, Day 1: Name: ____________________

Close Reading Guide. Eudoxus of Cnidos, 408–355 BCE: The Problem of Comparing Lengths

For Greek mathematicians of this era, whole numbers were everything (like 'atoms' today?). Assuming all pairs of lengths were commensurable, then each pair of lengths could be expressed as an integer multiple, say M and N, of some common unit length. Consequently, lengths be would proportional using ratios of integers (rationals), e.g., M/N. But we have seen that all lengths are NOT commensurable, in particular the edge of a square and its diagonal are not. So now what? Eudoxus was able to use proportionality (ratios) to talk about lengths, but it required an infinite number of comparisons (to every rational), not just one comparison. **Eudoxus' method of comparing lengths** rested on two principles:

1. the ability to compare two lengths, saying which is larger;
2. the ability to construct integer multiples of any length. (This is the so-called Archimedean Principle which we will encounter later and which is also important in Math 360, Foundations of Geometry.)

Exercise 1. Eudoxus compared two pairs of lengths (one or more of which might be irrational) as follows: Length a is to length b as c is to d if for every pair of whole numbers N and M, the length of the multiple Na compares to Mb in the same way as Nc compares to Md. Here "compares" means shorter, longer, or equal length. So Eudoxus is saying a is to b as c is to d if for each choice of $M, N \in \mathbb{N}$, exactly one of the following holds:

1. $Na = Mb$ if and only if $Nc = Md$
2. $Na < Mb$ if and only if $Nc < Md$
3. ____ Fill in.

Note: If a is to b as c is to d, then whether (1), (2), or (3) holds will depend on the choice of M and N. But no matter what choice we make, one of these statements will hold. On the other hand, if a is NOT to b as c is to d, then there should be a choice of M and N for which neither (1), nor (2), nor (3) holds. For example, we might have $Na < Mb$ but $Nc > Md$.

Exercise 2. Try this now:

a) Let $a = 3$, $b = 9$, $c = 4$, and $d = 12$. [Of course we want to say a is to b as c is to d.] If $M = 5$ and $N = 3$, does (1), (2), or (3) hold?

b) What if $M = 1$ and $N = 3$?

c) If $M = 2$ and $N = 7$?

d) Try some others. You should see that no matter how we choose $M, N \in \mathbf{N}$, either (1), (2), or (3) holds.

Exercise 3. Now try this: Let $a = 3$, $b = 6$, $c = 4$, and $d = 12$.

a) If $M = 5$ and $N = 3$, does (1), (2), or (3) hold?

b) If $M = 2$ and $N = 5$, does (1), (2), or (3) hold?

c) What would Eudoxus conclude?

Exercise 4: Translation. Take each of the three statements in Exercise 1 and divide the first (in)equality by Nb and the second by Nd. Eudoxus is saying: *The numbers a/b and c/d (which need not be rational) are equal if and only if for each rational number M/N one of the following cases holds:*

1. ____ or,
2. $a/b < M/N$ and $c/d < M/N$ or,
3. ____

If Statement 1 should happen to hold for some pair of integers N and M, then the number $a/b = c/d$ is actually rational. If a/b is not rational, then Statement 1 can never hold. But Statements 2 and 3 say a/b and c/d will be considered equal (whatever they are) if for any given rational M/N, either both a/b and c/d are smaller than M/N or both are larger. In other words, a/b and c/d **split the rationals in the same place**. There is no rational number M/N that is simultaneously larger than one and smaller than the other—no rational lies between a/b and

c/d. This means *an irrational quantity is completely determined by the way it divides (severs, cuts) the rationals.* Intuitively, then, the rationals are incomplete in some way and what lies between them are the 'irrationals.'

Exercise 5. Suppose x and y are two different 'irrationals.' Explain, using the material above, why Eudoxus would say there must be some rational number between them.

Concluding Remarks

> The comparison of the rational numbers with a straight line has led to the recognition of the existence of gaps, of a certain incompleteness or discontinuity of the rationals, while we ascribe to the straight line completeness, absence of gaps, or continuity. Dedekind (1831–1916)

How should we define numbers to fill these gaps? This is a bit vexing. We can't simply define the irrationals as the "gaps" since there is nothing there! 2000 years later, Dedekind had the following insight (see pages 14–16):

> I find the essence of continuity in the following principle: If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into these two classes, this severing of the straight line into two portions. Dedekind