Math 331 Homework: Day 4

Quote of the Day

> For brevity and in order not to weary the reader I suppress the proofs of these theorems which follow immediately from the definitions of the previous section. *Dedekind*

Review Section 1.2. Make sure to memorize the definition of **Dedekind cut**, **upper bound**, and **least upper bound**. Look ahead to the first part of Section 1.3 for the Field Axioms.

Hand In: Due Wednesday

Read all parts of all problems. I will add a few more problems to this set from Section 1.3.

1. Problem 1.2.3.

2. **Theorem:** Assume that λ is an upper bound for a subset S in $\mathbb{R}$. *λ is the least upper bound of S if and only if for any real number $\beta < \lambda$, there is a real number $\alpha \in S$ such that $\beta < \alpha$.* Aysmel, Abigael, Ruiqian, Stuart, Ryo prove $\Rightarrow$. Tianrui, Ivy, Jena, Caroline, Kelcie prove $\Leftarrow$.

3. Let α be a cut. By definition of cut, there is $s \in \mathbb{Q}$ so that $s \notin \alpha$.

- (a) Everyone: Prove: If $p \in \alpha$, then $p < s$.
- (b) Aysmel, Abigael, Ruiqian, Tianrui, Ivy: Determine whether $\alpha^C = \{r \in \mathbb{Q} : r \notin \alpha\}$ is a cut. Prove your answer. (I used part (a).)
- (c) Jena, Caroline, Kelcie, Stuart, Ryo: Determine whether $\alpha_- = \{-q : q \in \alpha\}$ is a cut. Prove your answer. (I used part (a).)

4. Problem 1.2.6. Problem 2 above may be helpful here.

5. Problem 1.2.7. Prove that the irrationals are dense. You may assume that $1 < \sqrt{2}$.

6. Problem 1.2.8. Greatest lower bounds in $\mathbb{R}$ can be viewed as the "mirror images" of least upper bounds.

7. (a) (Generalization) Let $(A, \leq)$ be any partially ordered set and let S be a subset of A. Define the terms **lower bound** of S and **greatest lower bound** of S.

- (b) Prove that if S has a greatest lower bound λ, then it is unique.

8. Problem 1.2.13. Jena, Caroline, Ruiqian, Tianrui, Ivy do part (a); Kelcie, Stuart, Ryo, Aysmel, Abigael do (b).

These extra credit problems may be done with a partner. (Hand in only 1 copy with with both your names.) Volunteer to present them!

9. **Bonus Challenge.** X be a non-empty set. Recall that $\mathcal{P}(X)$ is the power set of X, the set of all subsets of X. Consider the partially ordered set $(\mathcal{P}(X), \subseteq)$. Prove that every non-empty subset K of $\mathcal{P}(X)$ has a greatest lower bound (using your definition above). Hint: Consider what we did to find the least upper bound of K. What might you try for the greatest lower bound?

10. **Bonus Challenge.** Prove: If X is totally ordered, then X satisfies then the law of trichotomy. (See page 70 in *Chapter Zero*.)

11. **Bonus Challenge.** The rational numbers 1 and 2 are the real numbers (cuts) defined by $\mathbf{One} = \{q \in \mathbb{Q} \mid q < 1\}$ and $\mathbf{Two} = \{p \in \mathbb{Q} \mid p < 2\}$. Prove (using our definition of cut addition) that: $\mathbf{One} + \mathbf{One} = \mathbf{Two}$. Hint: Remember this a set equality you are trying to prove.

Answers

Problems

1. Suppose that $\lambda = \text{lub}(A)$. Let $B = \{ka \mid a \in A\}$, where $k > 0$.

(a) Show that $k\lambda$ is an upper bound for the set B.

Proof: Since λ is an upper bound for A, then $\forall a \in A$, we have $a \leq \lambda$. Since $k > 0$, then $\forall ka \in B$ we have $ka \leq k\lambda$. So $k\lambda$ is an upper bound for B.

(b) Show that $k\lambda$ is the least upper bound for B.

By contradiction: Suppose that γ were another upper bound for B with $\gamma < k\lambda$. Then $\forall ka \in B$ we have $ka \leq \gamma < k\lambda$. So $\forall a \in A$, we have $a \leq \frac{\gamma}{k} < \lambda$. But then γ/k is a smaller upper bound for A than λ which contradicts that λ is lub (A).

(c) What can happen if $k < 0$?

If $k < 0$ then $a \leq \lambda$ means $ka \geq k\lambda$ and $k\lambda$ is not an upper bound for B. The proof above fails.

2. **Theorem:** Assume that λ is an upper bound for a subset S in $\mathbb{R}$. *λ is the least upper bound of S if and only if for any real number $\beta < \lambda$, there is a real number $\alpha \in S$ such that $\beta < \alpha$.*

$\Rightarrow$ (by contradiction): Given $\lambda = \text{lub}(S)$. Assume that there is some number $\beta < \lambda$ and that there is *no* element α in S such that $\beta < \alpha$. Then $\forall \alpha \in S$ we have $\alpha \leq \beta$. So β is an upper bound for S and $\beta < \lambda$. This contradicts that λ is lub(S).

$\Leftarrow$ (by contradiction): Given: For any real number $\beta < \lambda$, there is a real number $\alpha \in S$ such that $\beta < \alpha$. Show $\lambda = \text{lub}(S)$. Let γ be another upper bound for S with $\gamma < \lambda$. Then by the given assumption, $\exists \alpha \in S$ so that $\gamma < \alpha$. This contradicts that γ is an upper bound for S.

3. Let α be a cut. By definition of cut, there is $s \in \mathbb{Q}$ so that $s \notin \alpha$.

(a) Prove: If $p \in \alpha$, then $p < s$.

Proof: Assume not. Then either $p = s$ (which contradicts that $p \in \alpha$ and $s \in \alpha^C$) or $p > s$. But if $p > s$ and α is a cut and $s \in \mathbb{Q}$, then by part (ii) of the definition of cut, $s \in \alpha$ which contradicts that $s \in \alpha^C$.

(b) Determine whether $\alpha^C = \{r \in \mathbb{Q} : r \notin \alpha\}$ is a cut. Prove your answer.

By contradiction: Assume α^C is a cut. From part (a) we know that $\exists s \in \alpha^C$ and $p \in \alpha$ with $p < s$. But this contradicts part (ii) of the definition of α^C being a cut since p is a rational that is both not in α^C and less than s, with $s \in \alpha^C$.

(c) Determine whether $\alpha_- = \{-q : q \in \alpha\}$ is a cut. Prove your answer.

By contradiction: Assume α_- is a cut. Since α is a cut, there is $s \in \mathbb{Q}$ so that $s \notin \alpha$. Then by definition, $-s \notin \alpha_-$. Now let $p \in \alpha$ (this implies that $-p \in \alpha_-$). Then by part (a), $p < s$. But $p < s$ implies $-s < -p$. So if α_- were a cut, then $-s \in \alpha_-$. Contradiction.

4. Let T and V be sets of real numbers with least upper bounds λ_1 and λ_2, respectively. Consider the set $S = \{\tau + \nu \mid \tau \in T,\ \nu \in V\}$. Prove that $\lambda_1 + \lambda_2 = \text{lub}(S)$.

To use problem 2, first show that $\lambda_1 + \lambda_2$ is <u>an</u> upper bound for S. Let $\sigma \in S$. Then $\sigma = \tau + \nu$ for some $\tau \in T$ and $\nu \in V$. Then $\tau \leq \lambda_1 = \text{lub}(T)$ and $\nu \leq \lambda_2 = \text{lub}(V)$, so $\sigma = \tau + \nu \leq \lambda_1 + \lambda_2$. Therefore, $\lambda_1 + \lambda_2$ is an upper bound for S.

Now use problem 2. Let β be any number smaller than $\lambda_1 + \lambda_2$, i.e., let $\beta < \lambda_1 + \lambda_2$. Show β is not an upper bound for S. Let $\delta = \lambda_1 + \lambda_2 - \beta$. Then $\delta > 0$ and $\beta = \lambda_1 + \lambda_2 - \delta$. By problem 3, since $\lambda_1 = \text{lub}(T)$ there is a $\tau' \in T$ so that $\tau' > \lambda_1 - \frac{\delta}{2}$ and a $\nu' \in V$ so that $\nu' > \lambda_2 - \frac{\delta}{2}$. Let $\sigma' = \tau' + \nu' \in S$. Then $\sigma = \tau' + \nu' > \lambda_1 - \frac{\delta}{2} + \lambda_2 - \frac{\delta}{2} = \beta$, so β is not an upper bound for S. Therefore, $\lambda_1 + \lambda_2 = \text{lub}(S)$.

5. Prove that the irrationals are dense. That is, if $\alpha < \beta$, then there is a rational t so that $\alpha < t < \beta$.

(a) Prove that if r and $s \neq 0$ are rational, then $r + s\sqrt{2}$ is irrational.

Proof: By contradiction. Assume $r + s\sqrt{2} = \frac{M}{N}$ is rational. Then since r and s are rational, we have

$$r + s\sqrt{2} = \frac{J}{K} + \frac{P}{Q}\sqrt{2} = \frac{M}{N}$$

where $J, K, P, Q \in \mathbb{Z}$ and $K, Q, N \neq 0$ and $P \neq 0$ since $s \neq 0$. But then solving for $\sqrt{2}$ we find

$$\sqrt{2} = \frac{QKM - QJN}{PNK}$$

is rational. This contradicts that $\sqrt{2}$ is irrational.

(b) Show that $\sqrt{2} < m(\beta - \alpha)$ for some positive integer m.

Proof: Since $\sqrt{2}$ and $\beta - \alpha$ are both positive, by the Archimedean Property, there is a positive integer m so that $\sqrt{2} < m(\beta - \alpha)$.

(c) Let n be the largest integer less than $m\alpha$. Show that $m\alpha < n + \sqrt{2} < m\beta$.

Proof: Since $\sqrt{2} < m(\beta - \alpha)$, then $m\alpha + \sqrt{2} < m\beta$. Since $n < m\alpha$, then $n + \sqrt{2} < m\alpha + \sqrt{2}$. But also since n is the *largest* integer less than $m\alpha$, then $m\alpha \leq n + 1$. Since $1 < \sqrt{2}$, then $m\alpha \leq n + 1 < n + \sqrt{2}$. So combining inequalities we have

$$m\alpha < n + \sqrt{2} < m\beta \Longrightarrow \alpha < \frac{m}{n} + \frac{1}{n}\sqrt{2} < \beta,$$

where $\frac{m}{n} + \frac{1}{n}\sqrt{2}$ is irrational by part (a).

6. Prove **The Greatest Lower Bound Property:** Every nonempty set of real numbers that has a lower bound has a greatest lower bound (glb).

(a) Let S be a nonempty set of real numbers and assume that γ is a lower bound for S. Define A to be the set of additive inverses of elements in S, that is, $A = \{-s \mid s \in S\}$. Show that $-\gamma$ is an *upper bound* for A.

Proof: If γ is a lower bound for S, then $\forall s \in S$ we have $\gamma \leq s$. Therefore $\forall s \in S$ we have $-\lambda \geq -s$, so $-\gamma$ is an upper bound for A.

(b) By the least upper bound property of the real numbers, since A has an upper bound $-\gamma$, then A must have a *least upper bound* λ.

(c) Show that $-\lambda$ is a *lower bound* for S... and at the same time we'll show show $-\lambda$ is glb (S).

Proof: Remember that γ was *any* lower bound for S and that $-\gamma$ is an upper bound for A. Since λ is the least upper bound for A, $\forall -s \in A$ (that is, $\forall s \in S$), we have

$$-s \leq \lambda \leq -\gamma.$$

Multiplying by -1 we get $\forall s \in S$,

$$\gamma \leq -\lambda \leq s.$$

The second part of the inequality shows that $-\lambda$ is a lower bound for S and the first part of the inequality shows that $-\lambda$ is the greatest lower bound since $-\lambda$ is at least as big as any other lower bound γ.

7. (a) (Generalization) Let $(A, \leq)$ be any partially ordered set and let S be a subset of A. Define the terms **lower bound** of S and **greatest lower bound** of S.

Definition: γ is a **lower bound** of S in A if for all $s \in S$ we have $\gamma \leq s$.

Definition: λ is a **greatest lower bound** of S in A if λ is a lower bound for S and for any lower bound γ of S $\gamma \leq \lambda$.

(b) Prove that if S has a greatest lower bound λ, then it is unique.

Definition: γ and λ are both greatest lower bounds of S. Then by definition, since λ is a greatest lower bound of S, we have $\gamma \leq \lambda$ in A. Similarly, since γ is a greatest lower bound of S, we have $\lambda \leq \gamma$ in A. Since A is a poset, then $\lambda = \gamma$. Hence the glb is unique.

8. (a) Prove that given any $\epsilon > 0$ there is some natural number n such that $\frac{1}{n} < \epsilon$.

Proof: Since ϵ and 1 are both positive, we may apply the Archimedean property (with $\alpha = \epsilon$ and $\beta = 1$). So there is a positive integer n so that $1 < n\epsilon$. Therefore, $\frac{1}{n} < \epsilon$.

(b) Prove the result in another way using the density of the rationals.

Proof: Since $0 < \epsilon$, and since the rationals are dense, there is a rational $\frac{m}{n}$ so that $0 < \frac{m}{n} < \epsilon$, where $m, n \in \mathbb{N}$. Since $1 \leq m$, then $\frac{1}{n} \leq \frac{m}{n} < \epsilon$.