Math 331 Homework: Day 5

Practice and Reading

Reread Section 1.3 carefully. This section should be more familiar to you than cuts.

Hand In Wednesday: The final part of the assignment

1. (*a*) Problem 1.3.5. Just list which of Axioms 1–11 fail (if any) and explain why very briefly.
   (*b*) Problem 1.3.8. Proceed as above.
2. We will soon be working with order relations in an axiomatic way. But we have also defined ordering with cuts. Let α, β, λ be cuts. Using the cut definitions of $+$ and $\leq$, prove: If $\alpha \leq \beta$, then $\alpha + \lambda \leq \beta + \lambda$. (Hint: Remember these inequalities actually refer to subset relations.)
3. (**Optional Bonus**: Good practice if you are struggling with cuts.) We will be working with fields in an axiomatic way. We can verify some of the axioms for $\mathbb{R}$ by using cuts. This problem shows there is an additive identity for $\mathbb{R}$. From Problem 6 from Day 2, the rational number 0 thought of as a real number is the cut defined by $\mathbf{Zero} = \{q \in \mathbb{Q} \mid q < 0\}$. Prove: If α is any cut, then $\mathbf{Zero} + \alpha = \alpha$. Hint: Remember this is a set equality you are trying to prove, so you must prove two things.
4. Proofs involving the least upper bounds and greatest lower bounds all work the same way. Here's a theorem with such proofs. **Just read it for practice.**

Density of the Rationals: Another Approach

THEOREM 1.0.1. Let x be any real number. Then there is an integer n such that

$$n \leq x < n+1.$$

PROOF. Let $A = \{n \in \mathbb{Z} : n \leq x\}$. First we will show (by contradiction) that A is not empty.

Assume, instead, that $A = \emptyset$. Let $B = \{n \in \mathbb{Z} : n > x\}$. Since $A = \emptyset$, then $B = \mathbb{Z}$. Notice that B is bounded below by x. So B has a greatest lower bound β. By definition of glb, $\beta + 1$ is not a lower bound of B. Consequently by definition of lower bound, there exists $m \in B$ such that $m < \beta + 1$. Subtracting 1, we see $m - 1 < \beta$. But $m - 1 \in \mathbb{Z} = B$, so β is not a lower bound for B. Contradiction.

So now we know that A is not empty. But A is bounded above (by x), so A has a least upper bound α. By definition of least upper bound, $\alpha - 1$ is not an upper bound for A. So there exists $m \in A$ such that $\alpha - 1 < m$. Adding 1, we see $\alpha < m + 1$. Since α is an upper bound for A, it follows that $m + 1 \notin A$ which means $x < m + 1$. But $m \in A$ so $m \leq x$. Therefore, $m \leq x < m + 1$, as desired.

THEOREM 1.0.2 (Density of the Rational Numbers). If α and β are real numbers with $\alpha < \beta$, then there is a rational number r such that $\alpha < r < \beta$.

PROOF. Let α an β be distinct real numbers with $\alpha < b$. Then $0 < \frac{1}{\beta - \alpha}$. By the previous theorem, there is an integer n so that

$$n \leq \frac{1}{\beta - \alpha} < n+1. \tag{1.1}$$

(Notice that $n+1$ is positive since $\frac{1}{\beta-\alpha}$ is positive.) Since $\frac{1}{\beta-\alpha} < n+1$, multiplying by $\beta - \alpha$ gives

$$1 < (n+1)(\beta - \alpha). \tag{1.2}$$

Again by the previous theorem, there is an integer m so that

$$m \le (n+1)\alpha < m+1. \tag{1.3}$$

Adding 1 to the first part of the inequality (1.3) gives

$$m+1 \le (n+1)\alpha + 1, \tag{1.4}$$

and using $1 < (n+1)(\beta - \alpha)$ from (1.2) in the last part of (1.4) yields

$$m+1 \le (n+1)\alpha + 1 < (n+1)\alpha + (n+1)(\beta - \alpha) = (n+1)\beta. \tag{1.5}$$

Combining the second part of (1.3) with (1.5) we get

$$(n+1)\alpha < m+1 < (n+1)\beta.$$

But $n+1 > 0$ since so

$$\alpha < \frac{m+1}{n+1} < \beta. \tag{1.6}$$

So $r = \frac{m+1}{n+1}$.