Math 331 Homework: Day 6

Practice and Reading

Reread Section 1.3 carefully. This section should be more familiar. Look ahead to Section 1.4 which discusses the Heine-Borel Theorem and the Balzano-Weierstrass Theorem. The names are those of great mathematicians. The theorems and their proofs are technical (and long) and likely very different from what you have seen in other courses. They will require sustained effort to understand.

In *A Tour of Calculus* read chapters 7, 8, 9 when you have a chance.

Class, Practice, Journal

1. These first problems should be relatively quick and easy.

a) Prove: if $a \neq 0$, then $(a^{-1})^{-1} = a$. We did this in class today. Can you give a proof without looking back? What principle does this illustrate? Try to prove the companion result: $-(-a) = a$.

b) Prove: If $a \neq 0$, then $-(a^{-1}) = (-a)^{-1}$.

c) Prove: $-a + (-b) = -(a + b)$. [We usually write: $-a - b = -(a + b)$.]

2. We have defined $a > b$ and $a < b$. Now define $a \geq b$ and $a \leq b$.

3. Define the set of **tropical numbers** $\mathbb{T}$ to be $\mathbb{R} \cup \{-\infty\}$ with "addition" defined as $x \oplus y = \max\{x, y\}$ and "multiplication" as $x \odot y = x + y$. Determine which axioms of a field are satisfied by $\mathbb{T}$.

The Field Axioms

Axioms for Addition:

1. Closure: For every a and b in $\mathbb{F}$, there is a unique number $a + b \in \mathbb{F}$.
2. Commutativity: For every a and b in $\mathbb{F}$, $a + b = b + a$.
3. Associativity: For every a, b, and c in $\mathbb{F}$, $a + (b + c) = (a + b) + c$.
4. Identity: There exists an element of $\mathbb{F}$ denoted by 0 such that for every a in $\mathbb{F}$, $a + 0 = a$.
5. Inverses: For every a in $\mathbb{F}$, there is a number $-a$ in $\mathbb{F}$ so that $a + (-a) = 0$.

Axioms for Multiplication:

6. Closure: For every a and b in $\mathbb{F}$, there is a unique number $a \cdot b$ (often written ab) in $\mathbb{F}$.
7. Commutativity: For every a and b in $\mathbb{F}$, $a \cdot b = b \cdot a$.
8. Associativity: For every a, b, and c in $\mathbb{F}$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
9. Identity: There exists an element of $\mathbb{F}$ denoted by 1 which does not equal 0, such that for every a in $\mathbb{F}$, $a \cdot 1 = a$.
10. Inverses: For every a in $\mathbb{F}$ with $a \neq 0$, there is a number a^{-1} in $\mathbb{F}$ such that $a \cdot (a^{-1}) = 1$.

Distributive Axiom:

11. For all a, b, and c in $\mathbb{F}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

The Order Axioms

12. Trichotomy: There exists a subset P of $\mathbb{F}$ such that for any number, a, exactly one of the following holds:
    i. $a = 0$;
    ii. $a \in P$, in which case we say $a > 0$;
    iii. $-a \in P$, in which case we say $a < 0$.

    P is called the set of **positive elements** of $\mathbb{F}$.
13. Additive Closure of P: If $a > 0$ and $b > 0$, then $a + b > 0$.
14. Multiplicative Closure of P: If $a > 0$ and $b > 0$, then $a \cdot b > 0$.