Math 331 Homework: Day 7

Note of the Day

Bernhard Bolzano: Lived from 1781 to 1848. Bolzano is said to have successfully freed calculus from the concept of the infinitesimal. He also gave examples of one-to-one correspondences between the elements of an infinite set A and the elements of a proper subset of A.

Practice, Reading, Class, and Journal

Reread Section 1.4 which discusses the Heine-Borel Theorem and the Balzano-Weierstrass Theorem.

1. Find sets of real numbers that satisfy the given condition, if possible. If impossible, explain why.

- **a)** A set with no upper upper bound.
- **b)** A finite set with no upper bound.
- **c)** An infinite set with an upper and lower bound.
- **d)** An infinite set with a lower bound and no upper bound.
- **e)** A set S with an upper bound, but no least upper bound.
- **f)** A set S with a least upper bound λ that is not an element of S.
- **g)** A set S with a least upper bound λ that is an element of S.

2. Consider these collections of open intervals. Which are covers of $S = (0, 1]$? Which have finite subcovers?

- **a)** $\{O_n\}_{n\in\mathbb{N}} = \left\{\left(\frac{1}{n}, \frac{n+1}{n}\right) \mid n \in \mathbb{N}\right\}$
- **b)** $\{T_n\}_{n\in\mathbb{N}} = \left\{\left(\frac{n-1}{4}, \frac{n+2}{4}\right) \mid n \in \mathbb{N}\right\}$
- **c)** $\{U_n\}_{n\in\mathbb{N}} = \{(-n, n) \mid n \in \mathbb{N}\}$
- **d)** $\{W_n\}_{n\in\mathbb{N}} = \{(n, n+1) \mid n \in \mathbb{Z}\}$
- **e)** $\{V\} = \{(-3, 15)\}$
- **f)** $\{P_a, P_b\} = \{(0, 1), (-17, \infty)\}$
- **g)** $\{Q_x\}_{x\in[0,2]} = \left\{\left(x - \frac{1}{3}, x + \frac{1}{3}\right) \mid x \in [0, 2]\right\}$

3. Determine which of these sets are bounded. Which are bounded above and which are bounded below? If a set is bounded above, find its least upper bound. If it is bounded below, find its greatest lower bound.

- **a)** $\mathbb{Z}$
- **b)** $\mathbb{Q}$
- **c)** $A = \{1/n^2 \mid n \in \mathbb{N}\}$
- **d)** $B = (0, 1)$
- **e)** $C = [0, 1]$
- **f)** $D = \{1, 2, 3\}$
- **g)** $X = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$
- **h)** $Y = [0, 1)$

4. Let $\{O_n\}_{n\in\mathbb{N}} = \{(\frac{1}{n}, 1 - \frac{1}{n}) \mid n \in \mathbb{N}\}$ and $\{P_n\}_{n\in\mathbb{N}} = \{(\frac{1}{n}, 1 + \frac{1}{n}) \mid n \in \mathbb{N}\}$.

- **a)** Is $\{P_n\}$ an open cover of $[0, 1]$? Is there a finite subcover?
- **b)** Explain why any open cover of D has a finite subcover.
- **c)** $\{P_n\}$ covers $B = (0, 1)$. Is there a finite subcover?
- **d)** Does $\{O_n\}$ cover $(0, 1)$? If so, is there a finite subcover?
- **e)** Show that $\{U_n\}_{n\in\mathbb{N}} = \{(-n, n) \mid n \in \mathbb{N}\}$ covers $S = [\frac{1}{2}, \infty)$, but there is no finite subcover.
- **f)** Does $\{Q_x\}_{x\in[0,2]} = \left\{\left(x - \frac{1}{3}, x + \frac{1}{3}\right) \mid x \in [0, 2]\right\}$ cover? Is there a finite subcover?

Hand In Wednesday

1. Define the set of **tropical numbers** $\mathbb{T}$ to be $\mathbb{R} \cup \{-\infty\}$. (Note $-\infty$ is meant to denote an element such that for all $x \in \mathbf{R}$, $-\infty < x$. "Addition" is defined as $x \oplus y = \max\{x, y\}$ and "multiplication" as $x \odot y = x + y$ is ordinary addition (where $-\infty + x = -\infty$). It turns out that field axioms 1–3, 6–11 are all satisfied. That leaves:

- **a)** Does field axiom 4 (additive identity) hold? Carefully explain/prove your result.
- **b)** Does field axiom 5 (additive inverses) hold? Carefully explain/prove your result.

2. Prove that the complex numbers $\mathbb{C}$ are not an ordered field. Hint: Is i in P?

3. a) Let $a \in \mathbb{F}$, an ordered field with $a \neq 0$. Prove $a^2 > 0$. (One method: Use two cases: $a > 0$ and $a < 0$.)

b) Consider the statement: "In a field $\mathbb{F}$, if $a = -a$, then $a = 0$." You might try to prove this by saying: "Since $a = -a$, then

$$a + a = a + (-a) = 0 \text{ so } 2a = 0 \text{ so } a = 0.$$

But this argument does not work since $a + a$ can equal 0 even if $a \neq 0$. Consider the two element set $\mathbb{Z}_2 = \{0, 1\}$ where the following tables define the operations of addition and multiplication. You should mentally check that $\mathbb{Z}_2$ is, in fact, a field. It is the smallest possible field, because Axiom 9 implies that all fields must have at least two elements. Notice that $1 \oplus 1 = 0$ so $1 = -1$ even though $1 \neq 0$. The statement: "If $a = -a$, then $a = 0$" is false in $\mathbb{Z}_2$

$\oplus$	0	1
0	0	1
1	1	0

$\odot$	0	1
0	0	0
1	0	1

However, **prove:** In an **ordered** field $\mathbb{F}$, if $a = -a$, then $a = 0$.

4. Problem 1.3.17. Let's split this up the parts:

a) Aysmel, Abigael, Ruiqian,Ivy, Ryo

b) Stuart, Tianrui, Caroline, Jena, Kelcie (you may need Problem 1.3.16 which we did in class).

c) Me. Prove: $|x| + |y| \geq |x - y|$. See below.

Proof: From part (b) we have $|x - y| - |y| \leq |(x - y) + y| = |x|$ so $|x - y| \leq |x| + |y|$.

d) All. Hint: First, show the question is equivalent to showing: $-|x-y| \leq |x|-|y| \leq |x-y|$. The second inequality is part (a). So all you have to do is prove the first inequality.

5. This is a variation on Problem 1.3.19 with strict inequalities to make it easier. (Use Theorem 1.3.8.)

a) If $a < b$ and $c \leq d$, then $c - b < d - a$. (All)

The rest of the parts follow immediately from previous parts. If you don't see how, ask me for a hint.

b) If $a < b < d$ and $a < c < d$, then $c - b < d - a$. (All)

c) If $a < b < d$ and $a < c < d$, then $b - c < d - a$. (Hobart)

d) If $a < b < d$ and $a < c < d$, then $|c - b| < d - a$. (Me)

Proof: From part (b) we have $c - b < d - a$. From part (c) we have $b - c < d - a$ so multiplying by -1 (use Theorem 1.3.9(b)), we obtain $-(b-c) > -(d-a)$. This is the same as $c-b > -(d-a)$ or $-(d - a) < c - b$. So stringing together the very first and last of all the inequalities in this argument,

$$-(d - a) < c - b < d - a.$$

By Theorem 1.3.12, $|c - b| < d - a$.

e) If $a < b < c < d$, then $c - b < d - a$. (Wm. Smith)

6. Problem 1.3.20. Use the triangle inequality or a variation for each part.

a) If $|a - m| < 1$, then $|a| < |m| + 1$. Aysmel, Abigael, Jena, Caroline, Ryo.

b) If $|a - m| < |m|/2$, then $|m|/2 < |a|$. (Hint: I used $|a - m| = |m - a|$, Thm 1.3.16 and 1.3.17.) Ruiqian, Kelcie, Stuart, Ivy, Tianrui

7. Consider four element set $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ where the following tables define the operations of addition and multiplication. Assume that addition and multiplication are associative and distributive. Explain how can you check commutativity very easily. Determine which, if any, of the other 8 axioms of a field are NOT satisfied.

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\odot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

8. Suppose that F is an ordered field. Then we know that $0 \neq 1$, so there are at least two elements in F. Define 2 to be the element $1+1$. Prove that $2 \neq 0$ and that $2 \neq 1$. Hence there are at least three distinct elements in F. (Make a conjecture about how many elements F must have.)

These next problems are all straightforward and are just a way to check that you have the concepts down.

9. Problem 1.4.1, but use the interval $[1, 5]$ instead.

10. Problem 1.4.2 (a and c). No proof, just make sure the covers work.

11. Problem 1.4.4 (a).

12. Problem 1.4.5

13. Problem 1.4.11. No need to show work. Just carefully read and use Definition 1.4.3

Extra credit

14. If you have never worked with the integers mod n, you might try this problem. If you have taken Math 375, this is pretty easy. Let $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. Figure out the addition and multiplication tables in analogy with $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_4$. Is $\mathbb{Z}_5$ a field? (You may assume the operations are associative and distributive.) If so, is it an ordered field? Make a conjecture about when $\mathbb{Z}_n$ is a field and when it is not.

15. In the last homework assignment, most of you used the fact that $\sqrt{2} > 1$. This is not entirely obvious working from our axioms and theorems, so prove it. *You may assume that* $\sqrt{2} > 0$. Hint: Try a proof by contradiction. Consider $\sqrt{2} - 1$ and its conjugate.

16. You may assume that the collection $\{O_n\}_{n \in \mathbb{N}} = \{(\frac{2}{n}, 2) \mid n \in \mathbb{N}\}$ is an open cover for the interval $(0, 1]$. Carefully prove that $\{O_\alpha\}_{\alpha \in A}$ has no finite subcover of $(0, 1]$.