Math 331 Homework: For Day 10

Quote of the Day

> One magnitude is said to be the *limit* of another magnitude when the second may approach the first within any given magnitude, however small, though the first magnitude may never exceed the magnitude it approaches. *D'Alembert*, 1765

> Given a variable quantity always smaller or greater than a proposed constant quantity; but which can differ from the latter by less than any proposed quantity however small; this constant quantity is called the *limit* in greatness or smallness of the variable quantity. *Simon L'Huilier*

The Limit Definition

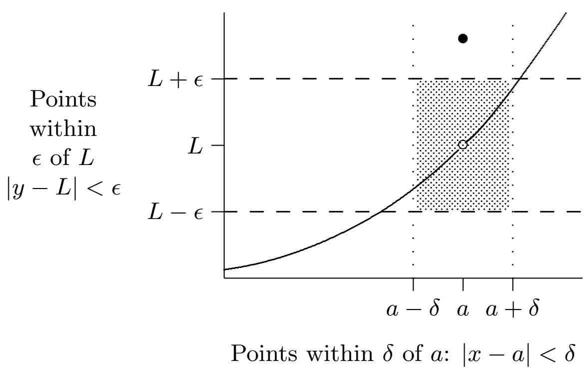
In Math 130 Calculus I we use:
Informal Definition of Limit: We say that $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to (but not equal to) a.

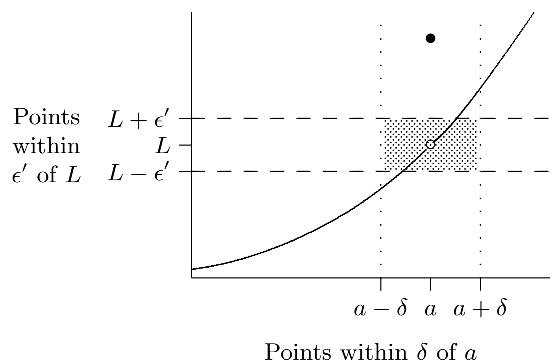
Compare this to the

Formal Definition of Limit: Let f be a function defined on some open interval containing a, except perhaps at a itself. We say that $\lim_{x\to a} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that

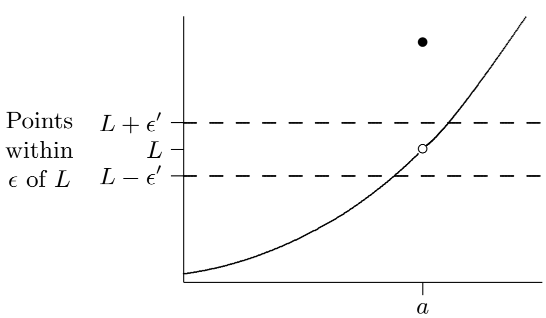
$$\text{if} \quad 0 < |x - a| < \delta, \quad \text{then} \quad |f(x) - L| < \epsilon.$$

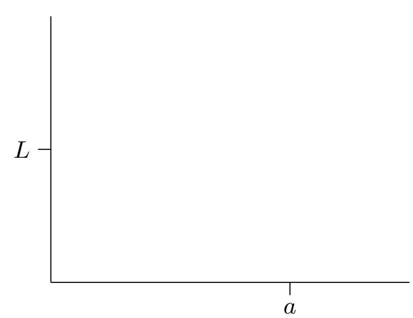
1. a) Figure on the left. For the ϵ that is given, the selected δ keeps $f(x)$ within the horizontal band, within ϵ of L over the interval from $a - \delta$ to $a + \delta$ (except perhaps at a). The shaded region consists of the points that satisfy both $|x - a| < \delta$ and $|f(x) - L| < \epsilon$.





b) Figure on the right. The same δ might not work for a different (smaller) ϵ'. Notice that f 'leaks out' of this narrower horizontal ϵ-band over the interval from $a-\delta$ to $a+\delta$. In other words $|f(x)-L| \geq \epsilon'$. However, can you find a smaller δ' that satisfies the limit definition? If so **draw** the new δ in the figure on the left below. The ability to do this—that is, for each $\epsilon > 0$ to find a corresponding $\delta > 0$ that satisfies the definition is what makes a limit "exist."





2. a) Because x is a variable, there is a 'hidden' universal quantifier in the limit definition. Here's a careful translation.

$$\forall \epsilon > 0, \exists \delta > 0 \text{ so that } \forall x, \text{ if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

Negate this statement. This negation is what it means for $\lim_{x \to a} f(x) \neq L$. This could mean either that the limit exists but is not L, or that the limit does not exist at all.

b) Draw a function $f(x)$ so that $\lim_{x \to a} f(x) \neq L$ in the picture on the right above.

3. The simplest non-constant function is $f(x) = x$. We know $\lim\limits_{x \to a} x = a$. Let's look at the definition: Given $\epsilon > 0$. How should **you choose** δ so that: if $0 < |x - a| < \delta$, then $|f(x) - f(a)| = |x - a| < \epsilon$.

4. $\lim\limits_{x \to 0} \sqrt{x}$ does not exist. Why not? There's a hypothesis that is not satisfied. Which?

5. Using the ϵ, δ definition of limit, show that these limits exist. The first two are relatively easy. They get harder. Use a common denominator in the last two.

a) $\lim\limits_{x \to -2} 2 - \frac{3}{2}x = 5$ **b)** $\lim\limits_{x \to 0} \dfrac{x}{x^2 + \frac{1}{2}} = 0$

c) $\lim\limits_{x \to 3} x^2 + x = 12$ **d)** $\lim\limits_{x \to 3} \dfrac{1}{1 + x} = \dfrac{1}{4}$ **e)** $\lim\limits_{x \to 2} \dfrac{1}{\sqrt{x + 7}} = \dfrac{1}{3}$

6. a) Let $f(x) = mx + b$ be a linear function. Show that for any real number a, $\lim\limits_{x \to a} f(x) = ma + b$. Hint: You will have to break this into two cases: (1) $m \neq 0$ and (2) $m = 0$.

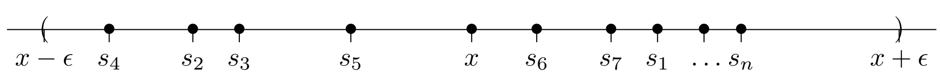
b) Explain why your answer to (a) solves Problems 2.2.10 and 11.

7. a) Assume that g is a bounded function, that is, $|g(x)| < B$ for all $x \neq 0$. Prove that $\lim\limits_{x \to 0} xg(x) = 0$.

b) Now redo Problem 5(b) in two sentences or less.

Hand In Monday (Test Wednesday)

1. Prove Lemma 1.4.5. which we discussed in class. This picture may help you think about the problem.



2. a) Suppose that λ is the least upper bound of a set S and that λ is *not* in S. Show that λ is an accumulation point of S. Hint: Use the definition of accumulation point and a property of lub's that you proved on a previous homework set. [Remember that $0 < |s - \lambda| < \epsilon$ means $\lambda - \epsilon < s < \lambda + \epsilon$ and $s \neq \lambda$.]

b) Give an example of a set S with a least upper bound λ that is *not* an accumulation point of S.

3. Problem 1.4.10. Hint: Use a previous problem on this set. This can be done in a couple of sentences.

4. Page 63 #2.2.1(f). If you can do this one you are in good shape. Use a preliminary bound.

5. Page 64 #2.2.9(a).

6. Prove: Assume that $\{O_\alpha\}_{\alpha \in A}$ is an open cover of the interval $[a, b]$ and that $f(x)$ is a function defined for all reals and hence on $[a, b]$. Assume further that f is **bounded** on each open set O_α (meaning that there is some number B_α so that for all $x \in O_\alpha$, we have $|f(x)| < B_\alpha$). Then f is actually bounded on the entire interval $[a, b]$ (meaning there is some number B so that for all $x \in [a, b]$, we have $|f(x)|$).