Math 331: Days 11 and 12

Class and Journal: Working with Limits

Last time we did #1 (a,b). The other easy ones are #2, 3 and 4. Any that we don't finish today we will work on next time in class. *Try at least a few of these before class and be prepared to discuss them.*

1. Using the ϵ, δ definition of limit, show that these limits exist. The first two are relatively easy. They get harder. Use a common denominator in the last two.

a) $\displaystyle\lim_{x\to -2} 2 - \tfrac{3}{2}x = 5$ **b)** $\displaystyle\lim_{x\to 0} \frac{x}{x^2 + \frac{1}{2}} = 0$

c) $\displaystyle\lim_{x\to 3} x^2 + x = 12$ **d)** $\displaystyle\lim_{x\to 3} \frac{1}{1+x} = \frac{1}{4}$ **e)** $\displaystyle\lim_{x\to 2} \frac{1}{\sqrt{x+7}} = \frac{1}{3}$

2. a) Let $f(x) = mx + b$ be a linear function. Show that for any real number a, $\lim_{x\to a} f(x) = ma + b$. Hint: You will have to break this into two cases: (1) $m \neq 0$ and (2) $m = 0$.

b) Explain why your answer to (a) solves Problems 2.2.10 and 11.

3. a) Assume that g is a bounded function, that is, $|g(x)| < B$ for all $x \neq 0$. Prove that $\lim_{x\to 0} xg(x) = 0$.

b) Now do Example 2.2.3 on page 59 in one sentence.

c) Now redo Problem 1(b) in two sentences or less.

4. Theorem: *If $\lim_{x\to a} f(x) = L$, then for every $\epsilon > 0$ there is $\delta > 0$ such that if x_1 and x_2 each satisfy $0 < |x - a| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$.* Justify each step in the following proof.

a) Given $\epsilon > 0$. Let $\epsilon' = \epsilon/2 > 0$. Explain why there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon'$.

b) Let x_1 and x_2 be any numbers that satisfy $0 < |x_i - a| < \delta$. Show that $|f(x_1) - f(x_2)| < \epsilon$. Hint: $|f(x_1) - f(x_2)| = |[f(x_1) - L] + [L - f(x_2)]|$. Now what key theorem about absolute values applies?

5. The Squeeze Theorem: *Assume that $f(x) \le g(x) \le h(x)$ for all x in some open interval containing a, except perhaps at a itself. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x)$ exists and equals L, also.* Justify each step in the following proof of the squeeze theorem.

a) Given $\epsilon > 0$ show that (explain why) there exists a $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \epsilon$.

b) Show that (explain why) if $0 < |x - a| < \delta_1$, then $L - \epsilon < f(x)$.

c) Show that (explain why) there exists a $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $h(x) < L + \epsilon$.

d) Let $\delta = \min\{\delta_1, \delta_2\}$. Show that if $0 < |x - a| < \delta$, then $L - \epsilon < g(x) < L + \epsilon$.

e) Complete the proof by showing that if $0 < |x - a| < \delta$, then $|g(x) - L| < \epsilon$.

f) Comment: Choosing δ to be the minima of several δ_i is a technique that we will often use.

6. a) Suppose that $f(x) \le 0$ for all x (except perhaps at $x = a$). Show that if $\lim_{x\to a} f(x) = L$, then $L \le 0$. (Hint: Assume instead that $L > 0$. Let $\epsilon = L/2$ and derive a contradiction.)

b) State and prove the analogue for $f(x) \ge 0$.

Practice

These include some of the problems above. Some will be assigned at a later date. This is meant to encourage you to carefully review and clean up your work above.

1. Page 63 #1 (c), (f), and (h).

2. Show that $\lim_{x\to a} |x| = |a|$. Hint: Use Problem 1.3.17.

3. The Squeeze Theorem above (or #7 on page 64).

4. Page 64 # 2.2.4 (a).