Topics for Test 1

Here is a list of some major topics with which every student in Math 331 should be familiar. The list is not exhaustive, but if you know all of this material you are well on your way to doing a good job on the exam.

1. **a)** **Definitions:** Rational and irrational number, Dedekind cut, upper and lower bound, least upper bound, greatest lower bound, accumulation point, density (dense set), limit.

- **b)** **Theorems:** Least Upper Bound Property, Archimedean Property, Density of the Rationals and Irrationals, The Triangle Inequality, the Heine-Borel Theorem, Bolzano-Weierstrass Theorem, "The Two Point Lemma."
- **c)** **Axioms:** Order axioms, least upper bound axiom, and be familiar with the first eleven field axioms.
- **d)** **Calculations:** Simple proofs of irrationality, working with the least upper bound and greatest lower bound definitions, working with simple Heine-Borel open cover situations, working with accumulation points, working with the triangle inequality, working with the Field Axioms to prove simple properties of real numbers. (E.g.: If $a \neq 0$, then $a^2 > 0$.). Limit calculations (including preliminary bounds). Calculations showing a limit does not exist.

Some Review and Practice Problems

1. Reprove Theorem 1.1.5: For any prime p, $\sqrt{p}$ is not rational. (You may assume that if m is a positive integer and p divides m^2, then p divides m).

2. **a)** Let $r \in \mathbb{R}$ and let n be an integer such that $n \geq 2$. Prove: If r^n is irrational, then r is irrational.

- **b)** Is the converse true? Prove it or give a counterexample.

3. Write down a careful definition of what it means for a set to be **closed** under an operation (e.g., under addition or multiplication). For each of the following sets determine whether or not it is closed under the operation given. If it is closed then write out a proof otherwise find an example to show that it is not closed.

- **a)** The set of numbers of the form $M + N\sqrt{3}$ where M and N are any integers under multiplication.
- **b)** The set of irrationals under multiplication.
- **c)** Let $\mathbb{F}$ be an ordered field and let P be the set of positive elements. Let N be the set of non-positive, non-zero elements in $\mathbb{F}$. In other words $N = \mathbb{F}(P \cup \{0\})$. (1) Prove that N is closed under addition. (2) Determine whether N is closed under multiplication. Hint: For each, think about Axiom 12.

4. Complete the following steps to show that $\sqrt{2} + \sqrt{3}$ is irrational. Use a proof by contradiction. Begin by assuming that $r = \sqrt{2} + \sqrt{3}$ is rational.

- **a)** Let $a = \sqrt{2} - \sqrt{3}$. Show that a must be irrational. Hint: Consider $r + a$
- **b)** Noting that r is rational and a is irrational write out the product ar to obtain a contradiction.

5. Prove **The Greatest Lower Bound Property:** Every nonempty set of real numbers that has a lower bound has a greatest lower bound (glb).

- **a)** Let S be a nonempty set of real numbers and assume that γ is a lower bound for S. Define A to be the set of additive inverses of elements in S, that is, $A = \{-s \mid s \in S\}$. Show that $-\gamma$ is an upper bound for A.
- **b)** Prove that A must have a least upper bound λ.
- **c)** Show that $-\lambda$ is the lower bound for S... and then that it is glb (S).

6. In the last homework assignment but one, most of you used the fact that $\sqrt{2} > 1$. This is not entirely obvious working from our axioms and theorems, so prove it. (You may assume $\sqrt{2} > 0$.)

7. Let $f(x) = x$ and let $g(x) = 1 - x^2$. (You may use calculus for these, if needed.)

- **a)** Let $A = \{f(x) | \, x \in [-1, 1]\}$. Find lub$(A)$. At which values of x does lub(A) occur?
- **b)** Let $B = \{g(x) | \, x \in [-1, 1]\}$. Find lub$(B)$. At which values of x does lub(B) occur?
- **c)** Let $C = \{f(x) + g(x) | \, x \in [-1, 1]\}$. Find lub$(C)$.
- **d)** Try Problem 1.2.1

8. Let $a \in \mathbb{F}$, where $\mathbb{F}$ is an ordered field and $a \neq 0$. If $a > 0$ prove that $a^{-1} > 0$.

9. Prove that the set of natural numbers $\mathbb{N}$ has no upper bound. (Contradiction?)

10. Let $\mathbb{F}$ is a field with $a, b \in \mathbb{F}$. Prove that $b^{-1} \cdot d^{-1} = (b \cdot d)^{-1}$.

11. The trichotomy axiom tells us that in an ordered field F, each number a fits into exactly one category: either $a = 0$, or $a > 0$, or $-a > 0$. Prove that $1 > 0$.

12. Suppose that $|x - y| < \frac{\epsilon}{2}$ and that $|y - z| < \frac{\epsilon}{2}$. Prove that $|x - z| < \epsilon$.

13. Consider the collection $\{O_x\} = \{(x-1, x+1) | x \in [0, 6]\}$ of open intervals.

a) How many sets does this collection have?

b) Find a finite subcover for $[0, 6]$ from this collection. What is the smallest possible number of sets in a such a subcover?

14. a) If $\{O_\alpha\}$ is an open cover of the union of intervals $[a_1, b_1] \cup [a_2, b_2]$. Prove that there is a finite subcover of $[a_1, b_1] \cup [a_2, b_2]$ from the sets $\{O_\alpha\}$.

b) Suppose that $n \in \mathbb{N}$ and that we $\{O_\alpha\}$ is an open cover of $\bigcup_{k=1}^{n} [a_k, b_k]$ Prove that there is a finite subcover of of $\bigcup_{k=1}^{n} [a_k, b_k]$ from the sets $\{O_\alpha\}$.

15. Use Definition 1.3.7 and Axiom 13 to prove the transitive property of inequalities; that is, if $a > b$ and $b > c$ then $a > c$.

16. For each of the following sets find three open covers (or explain why it is not possible): One that has a finite number of sets, one that has an infinite number of sets and also has a finite subcover, one that has no finite subcover.

a) $S = [-2, 4)$ **b)** $\mathbb{N}$ **c)** $L = [-3, 5]$ **d)** $T = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$

e) Determine the accumulation points of these same sets.

17. Show that $\lim_{x \to a} |x| = |a|$. Hint: Use Problem 1.3.17.

18. Let $f(x) = mx + b$ be a linear equation. Show that for any real number a, $\lim_{x \to a} f(x) = ma + b$.

19. More generally, review the limit calculations on recent handouts, Days 10 and 11 and the corresponding problems in the text: Problems 2.2.1, 2.2.4, 2.2.5.