Math 331 Homework: Day 14

Reading and Preparation for Discussion Next Class.

Read Sections 2.3 and 2.4, and look ahead to Section 2.5.

1. a) Assume that g is a bounded function for all $x \neq 0$, that is, $|g(x)| < B$ for all $x \neq 0$. Prove that $\lim_{x\to 0} xg(x) = 0$.

b) Now do Example 2.2.3 on page 57 in one sentence.

c) Now do Problem 2.2.5(a) on page 64 in one sentence. Hint: $F(x) = xD(x)$.

2. Page 70 # 2.3.3. Apply a theorem and a result about polynomials. Avoid an ϵ proof.

Hand In

Due Wednesday February 27. **You are encouraged to work with one partner.** If you do, hand in only one copy of the problems that you work on together.

1. (Practice w/bounding.) Prove: $\lim\limits_{x\to 2} x^3 = 8$ using the definition of limit (and not any of the theorems we have proven). If you don't remember how to factor $x^3 - 8$ then look it up in your calc book. You will eventually need to bound a quadratic using a preliminary bound. Use the triangle inequality to split it into three pieces and bound each piece separately to get an overall bound for the quadratic.

2. (Practice w/bounding.) Prove: $\lim\limits_{x\to 5} \frac{1}{2x-1} = \frac{1}{9}$ using the limit definition. Do not use a theorem.

3. On your last homework, you proved: If $f(x) \leq 0$ for all x (except perhaps at $x = a$) and if $\lim_{x\to a} f(x) = L$, then $L \leq 0$. Use this to help prove the following:

Suppose that $g(x) \leq h(x)$ for all x (except perhaps at a). If $\lim\limits_{x\to a} g(x) = M$ and $\lim\limits_{x\to a} h(x) = N$, prove that $M \leq N$. Hint: Let $f(x) = g(x) - h(x)$ and use a limit theorem. Avoid an ϵ proof.

4. Prove (using an ϵ and δ argument) Theorem 2.3.3, the constant multiple theorem. There are two cases to consider: $c \neq 0$ and $c = 0$.

5. Prove: For all $n \in \mathbb{N}$, if $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a}(f(x))^n = L^n$. Hint: Use a theorem and induction. Comment: This shows that $\lim_{x\to a} x^n = a^n$.

6. Prove the Squeeze Theorem (#8 on page 64).

7. On the exam you proved that $\lim_{x\to a} |x| = |a|$. Now prove this general result: If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} |f(x)| = |L|$. Caution: If your answer involves saying "choose $\delta = \epsilon$," you have made a mistake.

8. Page 71 # 2.3.5. No proofs necessary, just give the required functions, limits, and point a for each.

9. Prove: If $\lim\limits_{x\to a} f(x) = L$, then $\lim\limits_{x\to a} \sin(f(x)) = \sin L$.
Hints: Here are two well-known trig identities:

$$\sin(a) - \sin(b) = 2\cos(\tfrac{a+b}{2})\sin(\tfrac{a-b}{2}) \qquad \text{and} \qquad |\sin a| \leq |a|.$$

Use these to show $|\sin(f(x)) - \sin L| \leq |f(x) - L|$. Then complete the proof.

10. Give a proof of this theorem in three or fewer sentences using results we have previously proven.

Quotient Theorem. Assume that $\lim_{x\to a} f(x) = L$ and that $\lim_{x\to a} g(x) = M$ and $M \neq 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

11. Let $\{O_n\}_{n\in\mathbb{N}} = \{(n-1+\frac{(-1)^n}{n+1}, n+1+\frac{(-1)^n}{n+1}) : n \in \mathbb{N}\}$ is an open cover of $[0,5]$. Since $[0,5]$ is closed and bounded, by the H-B theorem this open cover must have a finite subcover. Find it and list all of the required intervals. (See the discussion below.)

Review of open covers

Can you give examples of open cover for the sets $[0,2]$, $(0,2)$ and $(5,\infty)$. For the latter two, create covers that do NOT have a finite subcover.

1. $\{O_n\}_{n\in\mathbb{N}} = \{(\frac{(-1)^n}{n}, 2+\frac{(-1)^n}{n}) : n \in \mathbb{N}\}$ is an open cover of $[0,2]$. Since $[0,2]$ is closed and bounded, by the H-B theorem this open cover must have a finite subcover, namely $\{(O_1, O_2\} = \{(-1,1), (\frac{1}{2}, \frac{5}{2})\}$. When we say a cover is finite, we mean that it has a finite number of sets (intervals) in it. Here the finite subcover contains only two sets.

2. $\{O_n\}_{n\in\mathbb{N}} = \{(\frac{1}{n}, 2-\frac{1}{n}) : n \in \mathbb{N}\}$ is an open cover of $(0,2)$ without a finite subcover.

   If a finite subcover C did exist, let M be the largest index of n for a set in the cover. Then, the greatest lower bound of C is $\frac{1}{M}$. At any rate, Cdoes not contain $\frac{1}{M+1}$, which is in $(0,2)$. Hence, C can't be a cover for $(0,2)$.

3. This is the same idea as 2; $\{(5+\frac{1}{n}, n) : n \in \mathbb{N}\}$ is an open cover of $(5,\infty)$ without a finite subcover. As above, it can be shown that given a finite union of open intervals from our cover (call it C), we can find an element arbitrarily close to 5 which is not in C; use the same argument as above.