Math 331 Day 15

Reading and Preparation for Discussion Next Class.

Re-read Section 2.4, and look ahead to Section 2.5 on Continuity. For pleasure, keep reading in *A Tour of the Calculus* chapters 12–14, p. 92–128.

Limit Summary. Assume that $\lim_{x\to a} f(x) = L$ and that $\lim_{x\to a} g(x) = M$. Let c be any constant.

0. $\lim\limits_{x\to a} c = c$.

1. $\lim\limits_{x\to a} cf(x) = cL$.

2. $\lim\limits_{x\to a} f(x) \pm g(x) = L \pm M$.

3. $\lim\limits_{x\to a} f(x)g(x) = LM$.

4. $\lim\limits_{x\to a} [f(x)]^n = M^n$ for any $n \in \mathbb{N}$.

5. If $M \neq 0$, then $\lim\limits_{x\to a} \dfrac{1}{g(x)} = \dfrac{1}{M}$ and $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.

6. $\lim\limits_{x\to a} |f(x)| = |L|$.

7. If $L > 0$, then $\lim\limits_{x\to a} \sqrt{f(x)} = \sqrt{L}$.

Practical Results. Some elementary results that follow.

1. $\lim\limits_{x\to a} mx + b = ma + b$.

2. $\lim\limits_{x\to a} x = a$.

3. $\lim\limits_{x\to a} x^n = a^n$ for any $n \in \mathbb{N}$.

4. Let $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots c_1 x + c_0$ be any polynomial. Then $\lim\limits_{x\to a} p(x) = p(a)$.

5. If $a > 0$, then $\lim_{x\to a} \sqrt{x} = \sqrt{a}$.

6. $\lim\limits_{x\to a} |x| = |a|$.

Continuity. Notice that in all of the practical examples above is that $\lim\limits_{x\to a} f(x) = f(a)$. The value of the limit is simply found by evaluating the function at the point in question. **This is absolutely not true for all limit calculations.** Nice functions such as those above are singled out. We say that f is **continuous** at $x = a$ if $\lim\limits_{x\to a} f(x) = f(a)$. See if you can prove the following using our results above (avoid an ϵ proof):

7. Let $r(x)$ be any rational function. If a is in the domain of r, then r is continuous at $x = a$.

8. Extra Credit: Let $F(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$. Prove: If $a \neq 0$, then $F(x)$ is not continuous at a.

OVER

Class Work on Other Types of Limits

1. The definition of limit says: Let f be defined on the open interval containing a, except perhaps at a itself. Then $\lim_{x\to a} f(x) = L$ if and only if for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$. Note: $0 < |x-a| < \delta$ means $a - \delta < x < a + \delta$ and $x \neq a$.

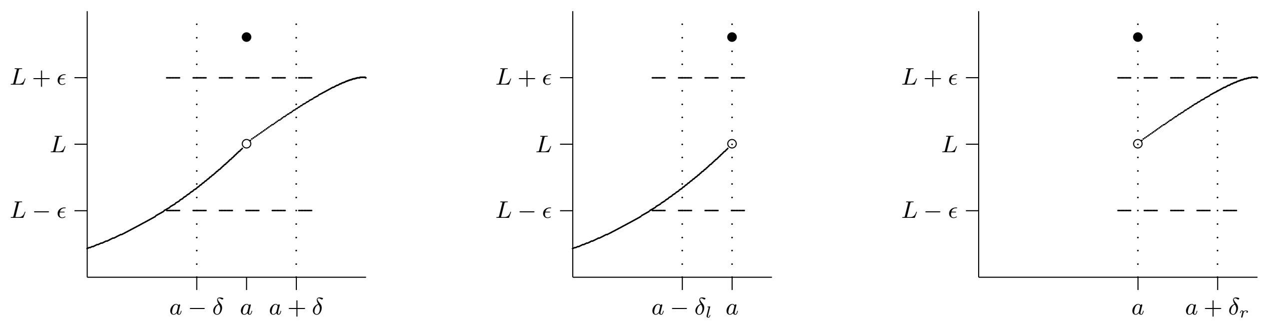
a) What is the only part of this definition that we need to modify to define $\lim_{x\to a^+} f(x) = L$?

b) To define $\lim_{x\to a^-} f(x) = L$?

c) Using the definition, prove that $\lim_{x\to 0^-} \sqrt{4-2x} = 0$.

2. Main Theorem: $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

Left Figure. Two-sided limit: For any $\epsilon > 0$, there exists $\delta > 0$ that keeps $f(x)$ within ϵ of L over the interval $(a-\delta, a+\delta)$, except perhaps at $x = a$. **Middle.** Limit from the left: For any $\epsilon > 0$, there exists $\delta_l > 0$ that keeps $f(x)$ within ϵ of L over the interval $(a - \delta_l, a)$. **Right.** Limit from the right: For any $\epsilon > 0$, there exists $\delta_r > 0$ that keeps $f(x)$ within ϵ of L over the interval $(a, a+\delta_r)$. Use the figures to help you see the proof.



a) Proof $\Rightarrow$: Given $\epsilon > 0$. Assume $\lim_{x\to a} f(x) = L$. You are required to find a δ_l for $\lim_{x\to a^-} f(x)$ and a δ_r for $\lim_{x\to a^+} f(x)$. Why is this easy to do and how do we 'get' these δ's?

b) Proof $\Leftarrow$: Given $\epsilon > 0$. Assume $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$. You are required to find a δ for $\lim_{x\to a} f(x)$. How do we 'get' this δ?

3. Let $f(x) = \begin{cases} x^2 + x - 2, & \text{if } x \leq 2, \\ \sqrt{5x+6} & \text{if } x > 2. \end{cases}$ Prove that $\lim_{x\to 2} f(x)$ exists. Avoid an ϵ proof. Hint: Use the theory on the other side of this sheet to show that each of the two-sided limits $\lim_{x\to 2} x^2 + x - 2$ and $\lim_{x\to 2} \sqrt{5x+6}$ exist. Then use the theorem above to show the one-sided limits $\lim_{x\to 2^-} x^2 + x - 2$ and $\lim_{x\to 2^+} \sqrt{5x+6}$ exist. Then use the theorem once again to show the two-sided limit $\lim_{x\to 2} f(x)$ exists.

4. a) Discuss the following: **Two-sided into One-sided Translation Principle:** Any two-sided limit theorem or statement can be translated into two one-sided limit theorems.

b) Give several examples.

5. The definition of limit says: Let f be defined on the open interval containing a, except perhaps at a itself. Then $\lim_{x\to a} f(x) = L$ if and only if for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

a) How must we adapt this definition to describe asymptotic behavior, that is, to define $\lim_{x\to+\infty} f(x) = L$?

b) What about $\lim_{x\to-\infty} f(x) = L$?

c) Use your definition to show that $\lim_{x\to\infty} \dfrac{x}{x^2+1} = 0$ or try $\lim_{x\to\infty} \dfrac{2x^2 - x + 2}{x^2+1} = 2$.

6. Curiously, 1-sided limits and a limits at infinity are related to each other. **T**heorem: Let f be defined on the open interval containing $(a, +\infty)$ and let $y = \frac{1}{x}$. Then $\lim_{x\to+\infty} f(x) = L$ if and only if $\lim_{y\to 0^+} f(\frac{1}{y}) = L$.

a) Give the proof in the $\Leftarrow$ direction.

b) XC: Give the proof in the $\Rightarrow$ direction.